arXiv:0705.2071v2 [math.QA] 6 Jun 2007

Tensor Product of the Fundamental Representations for the Quantum Loop Algebras of Type A at Roots of Unity

Yuuki Abe *
Graduate School of Mathematical Sciences,
University of Tokyo

Abstract

In this paper, we consider the necessary and sufficient conditions for the tensor product of the fundamental representations for the restricted quantum loop algebras of type A at roots of unity to be irreducible.

1 Introduction

Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra and let $\widetilde{\mathfrak{g}}$ be the loop algebra of $\mathfrak{g}$. Let q be an indeterminate and let $U_q(\widetilde{\mathfrak{g}})$ be the quantum algebra over $\mathbb{C}(q)$ associated with $\widetilde{\mathfrak{g}}$ and q, where $\mathbb{C}(q)$ is the rational function field. Let n be the rank of $\mathfrak{g}$ and let $I := \{1, 2, \cdots, n\}$ be the index set. For $\mathbf{a} \in \mathbb{C}(q)^\times$ and an index $\xi \in I$, there exists a finite-dimensional irreducible representation of $U_q(\widetilde{\mathfrak{g}})$ which is called a *fundamental representation* denoted by $\widetilde{V}_q(\pi_\xi^{\mathbf{a}})$ (see §4.4).

In 1997, Akasaka and Kashiwara gave the condition for $\widetilde{V}_q(\pi_{\xi_1}^{\mathbf{a}_1}) \otimes \cdots \otimes \widetilde{V}_q(\pi_{\xi_r}^{\mathbf{a}_r})$ to be irreducible in the case that $\widetilde{\mathfrak{g}}$ is of type $A_n^{(1)}$ and $C_n^{(1)}$ (see [AK]). In 2002, Varagnolo and Vasserot showed that condition in the simply laced case (see [VV]) and Kashiwara gave that condition for arbitrary $\widetilde{\mathfrak{g}}$ (see [K]). Moreover, in 2002, Chari showed the sufficient condition for the tensor product of the finite-dimensional irreducible representations of $U_q(\widetilde{\mathfrak{g}})$ to be a highest-weight representation (see [C]).

In particular, if $\widetilde{\mathfrak{g}} = A_n^{(1)}$, we explicitly obtain the necessary and sufficient conditions for $\widetilde{V}_q(\pi_{\xi_1}^{\mathbf{a}_1}) \otimes \cdots \otimes \widetilde{V}_q(\pi_{\xi_r}^{\mathbf{a}_r})$ to be irreducible. Indeed, we have the following theorem (see [AK] and Theorem 5.8 of this paper).

Theorem. *Let $m \in \mathbb{N}$, $\xi_1, \cdots, \xi_m \in I$, and $\mathbf{a}_1, \cdots, \mathbf{a}_m \in \mathbb{C}(q)^\times$. The following conditions (a) and (b) are equivalent.*

(a) $\widetilde{V}_q(\pi_{\xi_1}^{\mathbf{a}_1}) \otimes \cdots \otimes \widetilde{V}_q(\pi_{\xi_m}^{\mathbf{a}_m})$ is an irreducible representation of $U_q(A_n^{(1)})$.

(b) For any $1 \leq k \neq k' \leq m$ and $1 \leq t \leq \min(\xi_k, \xi_{k'}, n+1-\xi_k, n+1-\xi_{k'})$,

$$\frac{\mathbf{a}_{k'}}{\mathbf{a}_k} \neq q^{\pm(2t+|\xi_k-\xi_{k'}|)}.$$

We want to extend this theorem for the restricted quantm algebras of type $A_n^{(1)}$ at roots of unity.

Let l be an odd integer greater than 3, let ε be a primitive l-th root of unity, and let $U_\varepsilon^{\mathrm{res}}(\widetilde{\mathfrak{g}})$ be the restricted quantum algebra over $\mathbb{C}$ associated with $\widetilde{\mathfrak{g}}$ and ε (see [L89], [CP97], and §6.1). For a nonzero complex number $\mathbf{a}$ and an index $\xi \in I$, there exists a finite-dimensional irreducible representation of $U_\varepsilon^{\mathrm{res}}(\widetilde{\mathfrak{g}})$ which is called a *fundamental representation* denoted by $\widetilde{V}_\varepsilon^{\mathrm{res}}(\pi_\xi^{\mathbf{a}})$ (see §6.5).

In 1997, Chari and Pressley showed that for any finite-dimensional irreducible $U_\varepsilon^{\mathrm{res}}(\widetilde{\mathfrak{g}})$-representation V, there exist some nonzero complex numbers $\mathbf{a}_1, \cdots, \mathbf{a}_r$ and indexes $\xi_1, \cdots, \xi_r \in I$ such that V is isomorphic to a subquotient of $\widetilde{V}_\varepsilon^{\mathrm{res}}(\pi_{\xi_1}^{\mathbf{a}_1}) \otimes \cdots \otimes \widetilde{V}_\varepsilon^{\mathrm{res}}(\pi_{\xi_r}^{\mathbf{a}_r})$. However, the conditions for the irreducibility have not been given yet.

So we consider the conditions in the case that $\widetilde{\mathfrak{g}}$ is of type $A_n^{(1)}$. The main theorem is as follows (see Theorem 6.17):

*e-mail: abeyuuki@ms.u-tokyo.ac.jp

Theorem. *Let $m \in \mathbb{N}$, $\xi_1, \cdots, \xi_m \in I$, and $\mathbf{a}_1, \cdots, \mathbf{a}_m \in \mathbb{C}^\times$. The following conditions (a) and (b) are equivalent.*

(a) $\widetilde{V}_\varepsilon^{\rm res}(\pi_{\xi_1}^{\mathbf{a}_1}) \otimes \cdots \otimes \widetilde{V}_\varepsilon^{\rm res}(\pi_{\xi_m}^{\mathbf{a}_m})$ is an irreducible representation of $U_\varepsilon^{\rm res}(A_n^{(1)})$.

(b) For any $1 \leq k \neq k^{'} \leq m$ and $1 \leq t \leq \min(\xi_k, \xi_{k'}, n+1-\xi_k, n+1-\xi_{k'})$,

$$\frac{\mathbf{a}_{k'}}{\mathbf{a}_k} \neq \varepsilon^{\pm(2t+|\xi_k-\xi_{k'}|)}.$$

The organization of this paper is as follows. In §2, we fix some notations. In §3, we review the generic quantum algebras of type A_n and $A_n^{(1)}$. In §4, we introduce the fundamental representations of the generic quantum algebras of type A_n and $A_n^{(1)}$. In §5 (resp. §6, §7), we prove the main theorem for the generic quantum algebras of type $A_n^{(1)}$ (resp. the restricted quantum algebras of type $A_n^{(1)}$ at roots of unity, the small quantum algebras of type $A_n^{(1)}$).

2 Notations

We fix the following notations (see [Kac], [BK]). Let $\mathfrak{sl}_{n+1}$ be the finite-dimensional simple Lie algebra over $\mathbb{C}$ of type A_n. We define $I := \{1, 2, \cdots, n\}$. Let $(\mathfrak{a}_{i,j})_{i,j\in I}$ be the Cartan matrix of $\mathfrak{sl}_{n+1}$, that is, $\mathfrak{a}_{i,i} = 2$, $\mathfrak{a}_{i,j} = -1$ if $|i-j| = 1$, and $\mathfrak{a}_{i,j} = 0$ otherwise. Let $\{\alpha_i\}_{i\in I}$ (resp. $\{\alpha_i^\vee\}_{i\in I}$) be the set of the simple roots (resp. simple coroots) of $\mathfrak{sl}_{n+1}$ and let Δ (resp. Δ_+) be the root system (resp. the set of positive roots) of $\mathfrak{sl}_{n+1}$. Let $\mathfrak{h} = \bigoplus_{i\in I} \mathbb{C}\alpha_i^\vee$ be the Cartan subalgebra of $\mathfrak{sl}_{n+1}$ and let $\mathfrak{h}^* = \bigoplus_{i\in I} \mathbb{C}\alpha_i$ be the $\mathbb{C}$-dual space of $\mathfrak{h}$. We have a $\mathbb{C}$-bilinear map $\langle , \rangle : \mathfrak{h}^* \times \mathfrak{h} \longrightarrow \mathbb{C}$ such that $\langle \alpha_j, \alpha_i^\vee \rangle = \mathfrak{a}_{i,j}$ for any $i, j \in I$. Let $Q := \bigoplus_{i\in I} \mathbb{Z}\alpha_i$ (resp. $Q_+ := \bigoplus_{i\in I} \mathbb{Z}_+\alpha_i$) be the root lattice (resp. positive root lattice) of $\mathfrak{sl}_{n+1}$, where $\mathbb{Z}_+ := \{0, 1, 2, \cdots\}$. Let $\{\Lambda_i\}_{i\in I}$ be the fundamental weights of $\mathfrak{sl}_{n+1}$, that is,

$$\Lambda_i := \frac{1}{n+1}\{(n-i+1)\sum_{k=1}^{i} k\alpha_k + i\sum_{k=i+1}^{n}(n-k+1)\alpha_k\} \in \mathfrak{h}^*,$$

(see [H], §13). We have $\langle \Lambda_i, \alpha_j^\vee \rangle = \delta_{i,j}$ for any $i, j \in I$. Let $P := \bigoplus_{i\in I} \mathbb{Z}\Lambda_i$ (resp. $P_+ := \bigoplus_{i\in I} \mathbb{Z}_+\Lambda_i$) be the weight lattice (resp. positive weight lattice) of $\mathfrak{sl}_{n+1}$. Define a partial order $<$ in P whereby

$$\nu \leq \nu^{'} \text{ if and only if } \nu^{'} - \nu \in Q_+ \quad \text{for } \nu, \nu^{'} \in P. \tag{2.1}$$

Let $\widetilde{\mathfrak{sl}}_{n+1} = \mathfrak{sl}_{n+1} \otimes \mathbb{C}[t, t^{-1}]$ be the loop algebra of $\mathfrak{sl}_{n+1}$. We define $\widetilde{I} := I \sqcup \{0\}$ and

$$\mathfrak{a}_{0,0} := 2, \quad \mathfrak{a}_{i,0} := \mathfrak{a}_{0,j} := 0, \quad \mathfrak{a}_{n,0} := \mathfrak{a}_{0,n} := -1 \quad \text{for} \quad 1 \leq i, j < n.$$

Then $(\mathfrak{a}_{i,j})_{i,j\in\widetilde{I}}$ is the generalized Cartan matrix of $\widetilde{\mathfrak{sl}}_{n+1}$. Let $\{\alpha_i\}_{i\in\widetilde{I}}$ be the set of the simple roots of $\widetilde{\mathfrak{sl}}_{n+1}$. We define $\widetilde{\mathfrak{h}}^* := \mathbb{C}\alpha_0 \oplus \mathfrak{h}^*$. We have a symmetric $\mathbb{C}$-bilinear form $(,) : \widetilde{\mathfrak{h}}^* \times \widetilde{\mathfrak{h}}^* \longrightarrow \mathbb{C}$ such that $(\alpha_i, \alpha_j) = \mathfrak{a}_{i,j}$ for any $i, j \in \widetilde{I}$. Let s_i be the simple reflection on $\widetilde{\mathfrak{h}}^*$, that is, $s_i(\lambda) = \lambda - (\lambda, \alpha_i)\alpha_i$ for $\lambda \in \widetilde{\mathfrak{h}}^*$. The affine Weyl group $\widetilde{\mathcal{W}}$ of $\widetilde{\mathfrak{sl}}_{n+1}$ (resp. Weyl group $\mathcal{W}$ of $\mathfrak{sl}_{n+1}$) is generated by $\{s_i\}_{i\in\widetilde{I}}$ (resp. $\{s_i\}_{i\in I}$).

Let q be an indeterminate. For $r \in \mathbb{Z}$ and $m \in \mathbb{N} := \{1, 2, \cdots\}$, we define q-integers and Gaussian binomial coefficients in the rational function field $\mathbb{C}(q)$ whereby

$$[r]_q := \frac{q^r - q^{-r}}{q - q^{-1}}, \quad [m]_q! := [m]_q[m-1]_q \cdots [1]_q, \quad \left[\begin{array}{c} r \\ m \end{array}\right]_q := \frac{[r]_q[r-1]_q \cdots [r-m+1]_q}{[1]_q[2]_q \cdots [m]_q}.$$

Similarly, for $c \in \mathbb{C}\,(c \neq 0, \pm 1)$, we define

$$[r]_c := \frac{c^r - c^{-r}}{c - c^{-1}}, \quad [m]_c! := [m]_c[m-1]_c \cdots [1]_c, \quad \left[\begin{array}{c} r \\ m \end{array}\right]_c := \frac{[r]_c[r-1]_c \cdots [r-m+1]_c}{[1]_c[2]_c \cdots [m]_c}.$$

We define $[0]_q! := [0]_c! := 1$.

3 Quantum algebras : the generic case

3.1 Definitions and properties

Definition 3.1. Let $\widetilde{U}_q := U_q(\widetilde{\mathfrak{sl}}_{n+1})$ (resp. $U_q := U_q(\mathfrak{sl}_{n+1})$) be the associative $\mathbb{C}(q)$-algebra generated by $\{E_i, F_i, K_i^{\pm 1} \,|\, i \in \widetilde{I}$ (resp. $i \in I)\}$ with the following defining relations. We call $\widetilde{U}_q$ (resp. U_q) the quantum algebra of type $A_n^{(1)}$ (resp. A_n) or quantum loop algebra of type A_n:

$$\begin{aligned}
&K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad K_i K_j = K_j K_i, \quad K_0 = \prod_{i \in I} K_i^{-1},\\
&K_i E_j K_i^{-1} = q^{\mathfrak{a}_{i,j}} E_j, \quad K_i F_j K_i^{-1} = q^{-\mathfrak{a}_{i,j}} F_j,\\
&E_i F_j - F_j E_i = \delta_{i,j} \frac{K_i - K_i^{-1}}{q - q^{-1}},\\
&\sum_{p=0}^{1-\mathfrak{a}_{i,j}} (-1)^p E_i^{(p)} E_j E_i^{(1-\mathfrak{a}_{i,j}-p)} = \sum_{p=0}^{1-\mathfrak{a}_{i,j}} (-1)^p F_i^{(p)} F_j F_i^{(1-\mathfrak{a}_{i,j}-p)} = 0 \quad i \neq j,
\end{aligned}$$

for $i, j \in \widetilde{I}$ (resp. $i, j \in I$), where

$$E_i^{(m)} := \frac{1}{[m]_q!} E_i^m, \quad F_i^{(m)} := \frac{1}{[m]_q!} F_i^m \quad \text{for} \quad m \in \mathbb{N}. \tag{3.1}$$

Let U_q^+ (resp. U_q^-, U_q^0) be the $\mathbb{C}(q)$-subalgebra of U_q generated by $\{E_i\}_{i \in I}$ (resp. $\{F_i\}_{i \in I}$, $\{K_i^{\pm 1}\}_{i \in I}$). Then U_q has the triangular decomposition, that is, the multiplication defines an isomorphism of $\mathbb{C}(q)$-vector spaces:

$$U_q^- \otimes U_q^0 \otimes U_q^+ \xrightarrow{\sim} U_q, \tag{3.2}$$

(see [L90a]). It is well known that $\widetilde{U}_q$ (resp. U_q) has a Hopf algebra structure. The comultiplication Δ_H, counit ϵ_H, and antipode S_H of $\widetilde{U}_q$ (resp. U_q) are given by

$$\Delta_H(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta_H(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i, \quad \Delta_H(K_i) = K_i \otimes K_i, \tag{3.3}$$
$$\epsilon_H(E_i) = \epsilon_H(F_i) = 0, \quad \epsilon_H(K_i) = 1, \tag{3.4}$$
$$S_H(E_i) = -K_i^{-1} E_i, \quad S_H(F_i) = -F_i K_i, \quad S_H(K_i) = K_i^{-1}, \tag{3.5}$$

where $i \in \widetilde{I}$ (resp. I) (see [Ja], §3–§4 and [CP95], §2).

Let T_i be the $\mathbb{C}(q)$-algebra automorphism of $\widetilde{U}_q$ (resp. U_q) introduced by Lusztig in [L93], §37:

$$\begin{aligned}
&T_i(E_i^{(m)}) = (-1)^m q^{-m(m-1)} F_i^{(m)} K_i^m, \quad T_i(F_i^{(m)}) = (-1)^m q^{m(m-1)} K_i^{-1} E_i^{(m)},\\
&T_i(E_j^{(m)}) = \sum_{p=0}^{-m\mathfrak{a}_{i,j}} (-1)^{p-m\mathfrak{a}_{i,j}} q^{-p} E_i^{(-m\mathfrak{a}_{i,j}-p)} E_j^{(m)} E_i^{(p)} \quad (i \neq j),\\
&T_i(F_j^{(m)}) = \sum_{p=0}^{-m\mathfrak{a}_{i,j}} (-1)^{p-m\mathfrak{a}_{i,j}} q^{p} F_i^{(p)} F_j^{(m)} F_i^{(-m\mathfrak{a}_{i,j}-p)} \quad (i \neq j),\\
&T_i(K_j) = K_j K_i^{-\mathfrak{a}_{i,j}},
\end{aligned}$$

where $i, j \in \widetilde{I}$ (resp. $i, j \in I$) and $m \in \mathbb{N}$. For $w \in \widetilde{\mathcal{W}}$, let $w = s_{i_1} \cdots s_{i_m} (i_1, \cdots, i_m \in \widetilde{I}, m \in \mathbb{N})$ be a reduced expression of w. Then $T_w := T_{i_1} \cdots T_{i_m}$ is a well-defined automorphism of $\widetilde{U}_q$, that is, T_w does not depend on the choice of the reduced expression of w.

It is known that $\widetilde{U}_q$ has another realization which is called the Drinfel'd realization.

Definition 3.2 ([D])**.** Let $\mathcal{D}_q$ be an associative $\mathbb{C}(q)$-algebra generated by $\{X_{i,r}^{\pm}, H_{i,s}, K_i^{\pm 1} \mid i \in I, r, s \in \mathbb{Z}, s \neq 0\}$ with the defining relations

$$\begin{aligned}
&K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad [K_i, K_j] = [K_i, H_{j,s}] = [H_{i,s}, H_{j,s'}] = 0,\\
&K_i X_{j,r}^{\pm} K_i^{-1} = q^{\pm \mathfrak{a}_{i,j}} X_{j,r}^{\pm}, \qquad [H_{i,s}, X_{j,r}^{\pm}] = \pm \frac{[s\mathfrak{a}_{i,j}]_q}{s} X_{j,r+s}^{\pm},\\
&X_{i,r+1}^{\pm} X_{j,r'}^{\pm} - q^{\pm \mathfrak{a}_{i,j}} X_{j,r'}^{\pm} X_{i,r+1}^{\pm} = q^{\pm \mathfrak{a}_{i,j}} X_{i,r}^{\pm} X_{j,r'+1}^{\pm} - X_{j,r'+1}^{\pm} X_{i,r}^{\pm},\\
&[X_{i,r}^{+}, X_{j,r'}^{-}] = \delta_{i,j} \frac{\Psi_{i,r+r'}^{+} - \Psi_{i,r+r'}^{-}}{q - q^{-1}},\\
&\sum_{\pi \in \mathcal{S}_{\mathbf{m}}} \sum_{p=0}^{\mathbf{m}} (-1)^p \begin{bmatrix} \mathbf{m} \\ p \end{bmatrix}_q X_{i,r_{\pi(1)}}^{\pm} \cdots X_{i,r_{\pi(p)}}^{\pm} X_{j,r'}^{\pm} X_{i,r_{\pi(p+1)}}^{\pm} \cdots X_{i,r_{\pi(\mathbf{m})}}^{\pm} = 0, \quad (i \neq j),
\end{aligned}$$

for $r_1, \cdots, r_{\mathbf{m}} \in \mathbb{Z}$, where $\mathbf{m} := 1 - \mathfrak{a}_{i,j}$, $\mathcal{S}_{\mathbf{m}}$ is the symmetric group on $\mathbf{m}$ letters, and $\Psi_{i,r}^{\pm}$ are determined by

$$\sum_{r=0}^{\infty} \Psi_{i,\pm r}^{\pm} u^{\pm r} := K_i^{\pm 1} \exp(\pm (q - q^{-1}) \sum_{s=1}^{\infty} H_{i,\pm s} u^{\pm s}) \quad \text{in} \quad \mathcal{D}_q[[u]],$$

and $\Psi_{i,\pm r}^{\pm} := 0$ if $r < 0$.

Theorem 3.3 ([B])**.** *$\mathcal{D}_q$ is isomorphic to $\widetilde{U}_q$ as a $\mathbb{C}(q)$-algebra.*

The isomorphism from $\mathcal{D}_q$ to $\widetilde{U}_q$ is given in [B]. Here we introduce an isomorphism from $\widetilde{U}_q$ to $\mathcal{D}_q$ introduced in [CP94a], §2.5 (see also [AN], II-C). There exists a $\mathbb{C}(q)$-algebra isomorphism $T : \widetilde{U}_q \longrightarrow \mathcal{D}_q$ such that

$$\begin{aligned}
&T(E_i) = X_{i,0}^{+}, \quad T(F_i) = X_{i,0}^{-}, \quad T(K_i) = K_i,\\
&T(E_0) = (-1)^{m+1} q^{n+1} [X_{n,0}^{-}, \cdots [X_{m+1,0}^{-}, [X_{1,0}^{-}, \cdots [X_{m-1,0}^{-}, X_{m,1}^{-}]_{q^{-1}} \cdots]_{q^{-1}} \prod_{i \in I} K_i^{-1},\\
&T(F_0) = (-1)^{m+n} [X_{n,0}^{+}, \cdots [X_{m+1,0}^{+}, [X_{1,0}^{+}, \cdots [X_{m-1,0}^{+}, X_{m,-1}^{+}]_{q^{-1}} \cdots]_{q^{-1}} \prod_{i \in I} K_i,
\end{aligned}$$

for $m, i \in I$, where $[u, v]_{q^{\pm 1}} := uv - q^{\pm 1} vu$ for $u, v \in \widetilde{U}_q$ (T is independent of the choice of m). We identify $\widetilde{U}_q$ with $\mathcal{D}_q$ by this isomorphism.

Now, for $i \in I$ and $r \in \mathbb{Z}$, we define $\mathcal{P}_{i,r} \in \widetilde{U}_q$ whereby

$$\sum_{m=1}^{\infty} \mathcal{P}_{i,\pm m} u^m := \exp(-\sum_{s=1}^{\infty} \frac{q^s}{[s]_q} H_{i,s} u^s), \quad \mathcal{P}_{i,0} := 1, \quad \text{in} \quad \widetilde{U}_q[[u]]. \tag{3.6}$$

Let $\widetilde{U}_q^{\pm}$ (resp. $\widetilde{U}_q^{0}$) be the $\mathbb{C}(q)$-subalgebra of $\widetilde{U}_q$ generated by $\{X_{i,r}^{\pm} \mid i \in I, r \in \mathbb{Z}\}$ (resp. $\{K_i^{\pm 1}, \mathcal{P}_{i,r} \mid i \in I, r \in \mathbb{Z}\}$). Then $\widetilde{U}_q$ also has the triangular decomposition, that is, the multiplication defines an isomorphism of $\mathbb{C}(q)$-vector spaces:

$$\widetilde{U}_q^{-} \otimes \widetilde{U}_q^{0} \otimes \widetilde{U}_q^{+} \xrightarrow{\sim} \widetilde{U}_q, \tag{3.7}$$

(see [CP01], §4, [BCP], and [B]). We define

$$X_{\pm} := \sum_{i \in I, r \in \mathbb{Z}} \mathbb{C}(q) X_{i,r}^{\pm}, \quad X_{\pm}(i) := \sum_{j \in (I \setminus \{i\}), r \in \mathbb{Z}} \mathbb{C}(q) X_{j,r}^{\pm},$$

where $X_{\pm}(i) := 0$ if $I = \{i\}$.

Proposition 3.4 ([C], Proposition 2.2)**.** *Let $i \in I$, $r \in \mathbb{Z}$, and $s \in \mathbb{Z}^{\times}$.*

(a) Modulo $\widetilde{U}_q X_- \otimes \widetilde{U}_q (X_+)^2 + \widetilde{U}_q X_- \otimes \widetilde{U}_q X_+(i)$,

$$\Delta_H(X_{i,r}^+) = X_{i,r}^+ \otimes 1 + K_i \otimes X_{i,r}^+ + \sum_{p=1}^{r} \Psi_{i,p}^+ \otimes X_{i,r-p}^+ \quad \textit{if} \quad r \geq 0,$$

$$\Delta_H(X_{i,-r}^+) = K_i^{-1} \otimes X_{i,-r}^+ + X_{i,-r}^+ \otimes 1 + \sum_{p=1}^{r-1} \Psi_{i,-p}^- \otimes X_{i,-r+p}^+ \quad \textit{if} \quad r > 0.$$

(b) Modulo $\widetilde{U}_q (X_-)^2 \otimes \widetilde{U}_q X_+ + \widetilde{U}_q X_- \otimes \widetilde{U}_q X_+(i)$,

$$\Delta_H(X_{i,r}^-) = X_{i,r}^- \otimes K_i + 1 \otimes X_{i,r}^- + \sum_{p=1}^{r-1} X_{i,r-p}^- \otimes \Psi_{i,p}^+ \quad \textit{if} \quad r > 0,$$

$$\Delta_H(X_{i,-r}^-) = X_{i,-r}^+ \otimes K_i^{-1} + 1 \otimes X_{i,-r}^- + \sum_{p=1}^{r} X_{i,-r+p}^- \otimes \Psi_{i,-p}^- \quad \textit{if} \quad r \geq 0.$$

(c) Modulo $\widetilde{U}_q X_- \otimes \widetilde{U}_q X_+$,

$$\Delta_H(H_{i,s}) = H_{i,s} \otimes 1 + 1 \otimes H_{i,s}.$$

3.2 Representation theory of U_q and $\widetilde{U}_q$

Definition 3.5. Let V be a representation of $\widetilde{U}_q$ (resp. U_q).

(i) Let $v \in V$. If $X_{i,r}^+ v = 0$ for all $i \in I$, $r \in \mathbb{Z}$ (resp. $E_i v = 0$ for all $i \in I$), we call v a *pseudo-primitive vector* (resp. *primitive vector*) in V.

(ii) For any $\mu \in P$, we define

$$V_\mu := \{v \in V \mid K_i v = q^{\langle \mu, \alpha_i^\vee \rangle} v \quad \text{for all } i \in I\}.$$

If $V_\mu \neq 0$, we call V_μ a *weight space* of V. For $v \in V_\mu$, we call v a *weight vector* with weight μ and define $\mathrm{wt}(v) := \mu$.

(iii) For any $\mathbb{C}$-algebra homomorphism $\Lambda : \widetilde{U}_q^0 \longrightarrow \mathbb{C}(q)$, we define

$$V_\Lambda := \{v \in V \mid uv = \Lambda(u) v \quad \text{for all } u \in \widetilde{U}_q^0\}.$$

If $V_\Lambda \neq 0$, we call V_Λ a *pseudo-weight space* of V. For $v \in V_\Lambda$, we call v a *pseudo-weight vector* with *pseudo-weight* Λ and define $\mathrm{pwt}(v) := \Lambda$.

(iv) Let $\Lambda : \widetilde{U}_q^0 \longrightarrow \mathbb{C}$ be a $\mathbb{C}$-algebra homomorphism and λ be an element in P_+. If there exists a nonzero pseudo-primitive vector $v_\Lambda \in V_\Lambda$ (resp. primitive vector $v_\lambda \in V_\lambda$) such that $V = \widetilde{U}_q v_\Lambda$ (resp. $V = U_q v_\lambda$), we call V a *pseudo-highest weight representation* of $\widetilde{U}_q$ (resp. *highest-weight representation* of U_q) with the *pseudo-highest weight* Λ (resp. *highest weight* λ) generated by a *pseudo-highest weight vector* v_Λ (resp. *highest-weight vector* v_Λ).

Let V be a representation of $\widetilde{U}_q$ (resp. U_q). We call V of *type* **1** if

$$V = \bigoplus_{\mu \in P} V_\mu.$$

In general, finite-dimensional representations of $\widetilde{U}_q$ (resp. U_q) are classified into 2^n types according to $\{\sigma : Q \longrightarrow \{\pm 1\}$; group homomorphism$\}$. It is known that for any $\sigma : Q \longrightarrow \{\pm 1\}$, the category of finite-dimensional representations of $\widetilde{U}_q$ (resp. U_q) of type σ is essentially equivalent to the category of the finite-dimensional representations of $\widetilde{U}_q$ (resp. U_q) of type **1** (see [CP94b], §10–§11).

For any $\widetilde{U}_q$-representation (resp. U_q-representation) V, we have

$$X^{\pm}_{i,r}V_\mu \subset V_{\mu\pm\alpha_i}, \quad (\text{resp.} \quad E_iV_\mu \subset V_{\mu+\alpha_i}, \quad F_iV_\mu \subset V_{\mu-\alpha_i}), \tag{3.8}$$

where $i \in I$, $r \in \mathbb{Z}$, and $\mu \in P$.

The following theorem is well known (see [Ja]).

Theorem 3.6. *For any $\lambda \in P_+$, there exists a unique (up to isomorphism) finite-dimensional irreducible highest-weight U_q-representation $V_q(\lambda)$ with the highest weight λ of type* **1**. *Conversely, for any finite-dimensional irreducible U_q-representation V of type* **1**, *there exists a unique $\lambda \in P_+$ such that V is isomorphic to $V_q(\lambda)$ as a representation of U_q.*

We define a set of polynomials $\mathbb{C}(q)_0[t]$ whereby

$$\mathbb{C}(q)_0[t] := \{\pi(t) \in \mathbb{C}(q)[t]\,|\,\text{there exist some } a_1, \cdots, a_r \in \mathbb{C}(q) \text{ such that } \pi(t) = \prod_{s=1}^{r}(1-a_st)\}.$$

For any $\pi = (\pi_i(t))_{i\in I} \in (\mathbb{C}(q)_0[t])^n$, there exists a unique $\mathbb{C}(q)$-algebra homomorphism $\Lambda_\pi : \widetilde{U}^0_q \longrightarrow \mathbb{C}(q)$ such that

$$\Lambda_\pi(K_i^{\pm 1}) = q^{\pm \deg \pi_i(t)}, \quad \sum_{m=1}^{\infty}\Lambda_\pi(\mathcal{P}_{i,\pm m})t^m = \pi_i^{\pm}(t),$$

where

$$\pi_i^{+}(t) := \pi_i(t), \quad \pi_i^{-}(t) := \frac{t^{\deg \pi_i(t)}\pi_i(t^{-1})}{(t^{\deg \pi_i(t)}\pi_i(t^{-1}))|_{t=0}}. \tag{3.9}$$

From [CP97], §3, for $i \in I$, we have

$$\sum_{m=0}^{\infty}\Lambda_\pi(\Psi^{+}_{i,m})u^m = q^{\deg(\pi_i(u))}\frac{\pi_i(q^{-2}u)}{\pi_i(u)} = \sum_{m=0}^{\infty}\Lambda_\pi(\Psi^{-}_{i,-m})u^{-m} \quad \in \widetilde{U}_q[[u]], \tag{3.10}$$

in the sense that left- and right-hand terms are the Laurent expansions of the middle term about 0 and ∞, respectively (see also [CP95], Theorem 3.3 and [CP94b], 12.2B).

For any pseudo-highest weight representation of $\widetilde{U}_q$ with the pseudo-highest weight Λ_π, we simply call it a pseudo-highest representation of $\widetilde{U}_q$ with the pseudo-highest weight π.

Theorem 3.7 ([CP97], §2 and [C], §2)**.** *For any $\pi \in (\mathbb{C}(q)_0[t])^n$, there exists a unique (up to isomorphism) finite-dimensional irreducible pseudo-highest weight $\widetilde{U}_q$-representation $\widetilde{V}_q(\pi)$ with the pseudo-highest weight π of type* **1**.

From (3.2), (3.7), and (3.8), for $\lambda \in P_+$ and $\pi = (\pi_i(t))_{i\in I} \in (\mathbb{C}(q)_0[t])^n$, we have

$$\dim_{\mathbb{C}(q)}(V_q(\lambda)_\lambda) = 1, \quad V_q(\lambda) = \bigoplus_{\nu\le\lambda}V_q(\lambda)_\nu, \quad \dim_{\mathbb{C}(q)}(\widetilde{V}_q(\pi)_{\lambda(\pi)}) = 1, \quad \widetilde{V}_q(\pi) = \bigoplus_{\nu\le\lambda(\pi)}\widetilde{V}_q(\pi)_\nu, \tag{3.11}$$

where $<$ is the partial order defined in (2.1) and

$$\lambda(\pi) := \sum_{i\in I}\lambda_i(\pi)\Lambda_i := \sum_{i\in I}\deg(\pi_i(t))\Lambda_i \in P_+. \tag{3.12}$$

It is known that for any $\omega \in \mathcal{W}$ and $\mu \in P$,

$$\dim_{\mathbb{C}(q)}(V_q(\lambda)_{\omega\mu}) = \dim_{\mathbb{C}(q)}(V_q(\lambda)_\mu). \tag{3.13}$$

Proposition 3.8 ([CP95], Proposition 3.4)**.** *Let V (resp. V') be a pseudo-highest weight representations of $\widetilde{U}_q$ with the pseudo-highest weight π (resp. π') generated by a pseudo-highest weight vector v_π (resp. $v'_{\pi'}$). Then $v_\pi \otimes v'_{\pi'}$ is a pseudo-primitive vector with* $\mathrm{pwt}(v_\pi \otimes v'_{\pi'}) = \Lambda_{\pi\pi'}$.

Corollary 3.9 ([CP95], Corollary 3.5 and [CP94b], Theorem 12.2.6)**.** *Let $\pi, \pi' \in (\mathbb{C}(q)_0[t])^n$ and let v_π (resp. $v'_{\pi'}$) be a pseudo-highest weight vector in $\widetilde{V}_q(\pi)$ (resp. $\widetilde{V}_q(\pi')$). $\widetilde{V}_q(\pi\pi')$ is isomorphic to a quotient of the subrepresentation of $\widetilde{V}_q(\pi) \otimes \widetilde{V}_q(\pi')$ generated by $v_\pi \otimes v'_{\pi'}$. In particular, if $\widetilde{V}_q(\pi) \otimes \widetilde{V}_q(\pi')$ is irreducible, $\widetilde{V}_q(\pi) \otimes \widetilde{V}_q(\pi')$ is isomorphic to $\widetilde{V}_q(\pi\pi')$.*

3.3 The dual and involution representations of $\widetilde{U}_q$

For any $\widetilde{U}_q$-representation V, let $V^* = \{g : V \longrightarrow \mathbb{C}(q); \mathbb{C}(q)\text{-linear map}\}$ be the dual $\widetilde{U}_q$-representation of V. The action of $\widetilde{U}_q$ on V^* is defined by

$$(u.g)(v) := g(S_H(u).v), \quad \text{for } u \in \widetilde{U}_q,\ g \in V^*, \text{ and } v \in V,$$

where S_H is as in (3.5).

There exists a $\mathbb{C}(q)$-algebra involution $\Omega : \widetilde{U}_q \longrightarrow \widetilde{U}_q$ such that

$$\Omega(X^{\pm}_{i,r}) = -X^{\mp}_{i,-r}, \quad \Omega(H_{i,s}) = -H_{i,-s}, \quad \Omega(\Psi^{\pm}_{i,r}) = \Psi^{\mp}_{i,-r}, \quad \Omega(K_i^{\pm 1}) = K_i^{\mp 1}, \tag{3.14}$$

for $i \in I$, $r \in \mathbb{Z}$, and $s \in \mathbb{Z}^{\times}$ (see [CP96], Proposition 1.4(b)). For any $\widetilde{U}_q$-representation V, let V^{Ω} be the pull-back of V through Ω. For any finite-dimensional $\widetilde{U}_q$-representations V and W, we obtain

$$(V \otimes W)^* \cong W^* \otimes V^*, \quad (V \otimes W)^{\Omega} \cong W^{\Omega} \otimes V^{\Omega} \quad \text{as representations of } \widetilde{U}_q. \tag{3.15}$$

Proposition 3.10. *Let V be a finite-dimensional representation of $\widetilde{U}_q$. If V^* and V^{Ω} are pseudo-highest weight representations of $\widetilde{U}_q$, V is irreducible as a representation of $\widetilde{U}_q$.*

Proof. Let v_H be a pseudo-highest weight vector in V^{Ω} and let λ be the weight of v_H. Since V^{Ω} is a pseudo-highest weight representation, from (3.7), we have

$$V^{\Omega} = \widetilde{U}_q v_H = \mathbb{C}v_H \oplus (\bigoplus_{\mu < \lambda} V^{\Omega}_{\mu}).$$

Hence, from the definition of V^{Ω}, we have

$$V = \mathbb{C}v_H \oplus (\bigoplus_{\mu > -\lambda} V_{\mu}).$$

Let W be a proper $\widetilde{U}_q$-subrepresentation of V. We shall prove $W = 0$. Since V is generated by v_H as a representation of $\widetilde{U}_q$, v_H is not included in W. Hence we have

$$W \subset (\bigoplus_{\mu > -\lambda} V_{\mu}). \tag{3.16}$$

Now we define

$$\widetilde{W} := \{g \in V^* \mid g|_W = 0\},$$

and let g_H be an element in V^* defined by

$$g_H(v_H) := 1, \quad g(\bigoplus_{\mu > -\lambda} V_{\mu}) := 0.$$

Then, from (3.16), g_H is included in $\widetilde{W}$. Moreover the weight of g_H is maximam in U_q-representation V^*. Then, since V^* is a pseudo-highest weight $\widetilde{U}_q$-representation, g_H is a pseudo-highest weight vector in V^*. Thus we have

$$V^* = \widetilde{U}_q g_H \subset \widetilde{W}.$$

Therefore $W = 0$. □

Proposition 3.11 ([CP96], Proposition 5.1 and [C], (2.20), (2.21)). *Let $\pi = (\pi_i(t))_{i \in I} \in (\mathbb{C}(q)_0[t])^n$.*

(a) There exists an integer $c \in \mathbb{Z}$ depending only on $\mathfrak{sl}_{n+1}$ such that

$$V_q(\pi)^* \cong V_q(\pi_n(q^c t), \cdots, \pi_1(q^c t)).$$

(b) There exists a nonzero complex number $\kappa \in \mathbb{C}^{\times}$ depending only on $\mathfrak{sl}_{n+1}$ such that

$$V_q(\pi)^{\Omega} \cong V_q(\pi_n^-(q^2 \kappa t), \cdots, \pi_1^-(q^2 \kappa t)),$$

where $\pi_i^-(t)$ is as in (3.9).

3.4 The extremal vectors in $\widetilde{U}_q$

In this subsection, we introduce the extremal vectors in $\widetilde{V}_q(\pi)$ (see [C], §4 and [AK]). For $w \in W$, let $w = s_{i_1} s_{i_2} \cdots s_{i_k}$ be a reduced expression of w. For $\lambda = \sum_{i \in I} \lambda_i \Lambda_i \in P_+$, we define

$$m_{i_j}^{\lambda} := \langle s_{i_{j+1}} s_{i_{j+2}} \cdots s_{i_k} \lambda, \alpha_{i_j}^{\vee} \rangle,$$

where $m_{i_k}^{\lambda} := \lambda_{i_k}$. Then we have $m_{i_j}^{\lambda} \geq 0$ for all $1 \leq j \leq k$.

Definition 3.12. For $\pi = (\pi_i(t))_{i \in I} \in (\mathbb{C}(q)_0[t])^n$, let v_π be a pseudo-highest weight vector in $\widetilde{V}_q(\pi)$ and let $\lambda(\pi)$ be as in (3.12). For $w \in W$, let $w = s_{i_1} s_{i_2} \cdots s_{i_k}$ be a reduced expression of w. For $1 \leq j \leq k$, we define

$$v_\pi(s_{i_j} s_{i_{j+1}} \cdots s_{i_k}) := F_{i_j}^{(m_{i_j}^{\lambda(\pi)})} F_{i_{j+1}}^{(m_{i_{j+1}}^{\lambda(\pi)})} \cdots F_{i_k}^{(m_{i_k}^{\lambda(\pi)})} v_\pi,$$

which is called a *extremal vector* in $\widetilde{V}_q(\pi)$.

We have

$$X_{i_j,r}^{+} v_\pi(s_{i_{j+1}} s_{i_{j+2}} \cdots s_{i_k}) = 0, \tag{3.17}$$

for any $1 \leq j \leq k$ and $r \in \mathbb{Z}$. For $i, j \in I$ such that $i \leq j$, we define

$$\omega_{i,j} := (s_i s_{i+1} \cdots s_j)(s_1 s_2 \cdots s_{j-1})(s_1 s_2 \cdots s_{j-2}) \cdots (s_1 s_2) s_1. \tag{3.18}$$

Then $\omega_{1,n}$ is the longest element in $\mathcal{W}$. For any $\lambda = \sum_{i \in I} \lambda_i \Lambda_i \in P_+$, we have

$$\langle \omega_{i,j} \lambda, \alpha_{i-1}^{\vee} \rangle = \lambda_{j-i+2} + \lambda_{j-i+3} + \cdots + \lambda_j, \quad \text{if} \quad i \neq 1, \tag{3.19}$$

$$\langle \omega_{1,j} \lambda, \alpha_{j+1}^{\vee} \rangle = \lambda_1 + \lambda_2 + \cdots + \lambda_{j+1}. \tag{3.20}$$

Hence we obtain

$$v_\pi(s_{i-1} \omega_{i,j}) = F_{i-1}^{(\lambda_{j-i+2}(\pi) + \cdots + \lambda_j(\pi))} v_\pi(\omega_{i,j}), \quad \text{if} \quad i \neq 1, \tag{3.21}$$

$$v_\pi(s_{j+1} \omega_{1,j}) = F_{j+1}^{(\lambda_1(\pi) + \cdots + \lambda_{j+1}(\pi))} v_\pi(\omega_{1,j}). \tag{3.22}$$

Proposition 3.13 ([C], Proposition 4.1 and Proposition 6.3 (i))**.** *For $\pi \in (\mathbb{C}(q)_0[t])^n$, let v_π be a pseudo-highest vector in $\widetilde{V}_q(\pi)$. For $i, j \in I$ such that $i \leq j$, we have*

$$H_{i-1,1} v_\pi(\omega_{i,j}) = \sum_{k=j-i+2}^{j} q^{2j-i-k+1} \Lambda_\pi(H_{k,1}) v_\pi(\omega_{i,j}) \quad (i \neq 1),$$

$$H_{j+1,1} v_\pi(\omega_{1,j}) = \sum_{k=1}^{j+1} q^{j+1-k} \Lambda_\pi(H_{k,1}) v_\pi(\omega_{1,j}).$$

4 The fundamental representations: the generic case

4.1 The fundamental representations of U_q

For $\lambda \in P_+$, we call λ *minimal weight* if $\mu \in P_+$, $\mu \leq \lambda$ implies that $\mu = \lambda$. Moreover, we call a nonzero minimal weight a *minuscule weight* (see [Ja], §5A.1).

Proposition 4.1 ([H], §13, Exercises 13)**.** *Let $\lambda \in P_+$.*

(a) λ is minimal weight if and only if $\langle \lambda, \alpha^{\vee} \rangle \in \{0, \pm 1\}$ for all $\alpha \in \Delta$.

(b) For $\xi \in I$, Λ_ξ is a minuscule weight. Conversely, if λ is a minuscule weight, there exists an index $\xi \in I$ such that $\lambda = \Lambda_\xi$.

For $\xi \in I$, we call $V_q(\Lambda_\xi)$ a *fundamental representation* of U_q. We can construct these representations as follows. Let $W_q(\Lambda_\xi)$ be a $\#(\mathcal{W}\Lambda_\xi)$-dimensional $\mathbb{C}(q)$-vector space and let $\{z_\mu \mid \mu \in \mathcal{W}\Lambda_\xi\}$ be a $\mathbb{C}(q)$-basis of $W_q(\Lambda_\xi)$.

Proposition 4.2 ([Ja], §5A.1)**.** *(a) We can define a U_q-representation structure on $W_q(\Lambda_\xi)$ by the following formula:*

$$K_i^{\pm 1} z_\mu = q^{\pm(\mu,\alpha_i)} z_\mu, \tag{4.1}$$

$$E_i z_\mu = \begin{cases} z_{\mu+\alpha_i}, & \langle \mu, \alpha_i^\vee \rangle = -1, \\ 0, & \textit{otherwise}, \end{cases} \tag{4.2}$$

$$F_i z_\mu = \begin{cases} z_{\mu-\alpha_i}, & \langle \mu, \alpha_i^\vee \rangle = 1, \\ 0, & \textit{otherwise}, \end{cases} \tag{4.3}$$

for any $i \in I$ and $\mu \in \mathcal{W}\Lambda_\xi$.

(b) $W_q(\Lambda_\xi)$ is isomorphic to $V_q(\Lambda_\xi)$ as a representation of U_q.

We identify $W_q(\Lambda_\xi)$ with $V_q(\Lambda_\xi)$. Obviously, z_{Λ_ξ} is a highest-weight vector in $W_q(\Lambda_\xi)$. From (3.11) and (3.13), for any $\mu \in \mathcal{W}\Lambda_\xi$, we have

$$\dim_{\mathbb{C}(q)}(V_q(\Lambda_\xi)_\mu) = 1. \tag{4.4}$$

4.2 Another realization of the fundamental representations of U_q

We define $\widehat{I} := I \sqcup \{n+1\} = \{1, 2, \cdots, n+1\}$. For $\xi \in I$, we define

$$\mathcal{J}_\xi := \{J = \{j_1, j_2, \cdots, j_\xi\} \subset \widehat{I} \mid j_1 < j_2 < \cdots < j_\xi\}.$$

Let $L_q(\Lambda_\xi)$ be a $\#(\mathcal{J}_\xi)$-dimensional $\mathbb{C}(q)$-vector space. We regard $\mathcal{J}_\xi$ as a $\mathbb{C}(q)$-basis of $L_q(\Lambda_\xi)$.

Proposition 4.3 ([DO], §2.2 and [AK], B.1)**.** *(a) For $\xi \in I$, we can define a U_q-representation structure on $L_q(\Lambda_\xi)$ by the following formula: For $i \in I$ and $J \in \mathcal{J}_\xi$,*

$$E_i J = \begin{cases} (J \backslash \{i+1\}) \sqcup \{i\}, & \textit{if } i+1 \in J \textit{ and } i \notin J, \\ 0, & \textit{otherwise}, \end{cases} \tag{4.5}$$

$$F_i J = \begin{cases} (J \backslash \{i\}) \sqcup \{i+1\}, & \textit{if } i \in J \textit{ and } i+1 \notin J, \\ 0, & \textit{otherwise}, \end{cases} \tag{4.6}$$

$$K_i J = q^{\delta(i \in J) - \delta(i+1 \in J)} J, \tag{4.7}$$

where, for a statement θ,

$$\delta(\theta) := \begin{cases} 1, & \textit{if } \theta \textit{ is true}, \\ 0, & \textit{if } \theta \textit{ is false}. \end{cases} \tag{4.8}$$

(b) $L_q(\Lambda_\xi)$ is isomorphic to $V_q(\Lambda_\xi)$ as a representation of U_q.

For $\xi \in I$, we define

$$J_H^\xi := \{1, 2, \cdots, \xi\}. \tag{4.9}$$

Obviously, J_H^ξ is a highest-weight vector in $L_q(\Lambda_\xi)$.

4.3 The evaluation representations of $\widetilde{U}_q$

In this subsection, we introduce the evaluation representations of $\widetilde{U}_q$ to consider the fundamental representations of $\widetilde{U}_q$ in the next subsection.

Definition 4.4. The *extended quantum algebra* $U_q^{'} := U_q^{'}(\mathfrak{sl}_{n+1})$ is an associative $\mathbb{C}(q)$-algebra generated by $\{E_i, F_i, K_\mu^{'} \,|\, i \in I, \mu \in P\}$ with the defining relations

$$K_\mu^{'} K_\nu^{'} = K_{\mu+\nu}^{'}, \quad K_0^{'} = 1, \quad K_\mu^{'} E_j K_{-\mu}^{'} = q^{\langle \mu, \alpha_i^\vee \rangle} E_j, \quad K_\mu^{'} F_j K_{-\mu}^{'} = q^{-\langle \mu, \alpha_i^\vee \rangle} F_j,$$
$$E_i F_j - F_j E_i = \delta_{i,j} \frac{K_{\alpha_i}^{'} - K_{-\alpha_i}^{'}}{q - q^{-1}},$$
$$\sum_{r=0}^{1-\mathfrak{a}_{i,j}} (-1)^r E_i^{(r)} E_j E_i^{(1-\mathfrak{a}_{i,j}-r)} = \sum_{r=0}^{1-\mathfrak{a}_{i,j}} (-1)^r F_i^{(r)} F_j F_i^{(1-\mathfrak{a}_{i,j}-r)} = 0 \quad i \neq j,$$

for $i, j \in I$, $\mu, \nu \in P$.

Remark 4.5. Let V be a representation of U_q. For $i \in I$ and $\mu = \sum_{k \in I} \mu_k \Lambda_k \in P$, we define

$$c_{i,\mu} := \frac{1}{n+1} \{(n-i+1) \sum_{k=1}^{i} k\mu_k + i \sum_{k=i+1}^{n} (n-k+1)\mu_k\},$$

(see (2.1)). We can regard V as a representation of $U_q^{'}$ by using the following formula: for $v \in V_\mu$,

$$K_{\Lambda_i}^{'} v := q^{c_{i,\mu}} v.$$

Now, for $X \in \{E, F\}$, we define

$$X_\theta^+ := [X_n, [X_{n-1}, \cdots, [X_2, X_1]_{q^{-1}} \cdots]_{q^{-1}}, \quad X_\theta^- := [X_1, [X_2, \cdots, [X_{n-1}, X_n]_{q^{-1}} \cdots]_{q^{-1}},$$

where $[u, v]_{q^{\pm 1}} = uv - q^{\pm 1} vu$ for any $u, v \in U_q^{'}$.

Proposition 4.6 ([Ji], §2 and [CP94a], Proposition 3.4)**.** *For any $\mathbf{a} \in \mathbb{C}(q)^\times$, there exist $\mathbb{C}(q)$-algebra homomorphisms $\mathrm{ev}_{\mathbf{a}}^+$, $\mathrm{ev}_{\mathbf{a}}^- : \widetilde{U}_q \longrightarrow U_q^{'}$ such that*

$$\mathrm{ev}_{\mathbf{a}}^\pm(E_i) = E_i, \quad \mathrm{ev}_{\mathbf{a}}^\pm(F_i) = F_i, \quad \mathrm{ev}_{\mathbf{a}}^\pm(K_i) = K_{\alpha_i}^{'}, \quad \textit{for} \quad i \in I,$$
$$\mathrm{ev}_{\mathbf{a}}^\pm(E_0) = \mathbf{a} K_{\pm(\Lambda_1 - \Lambda_n)}^{'} F_\theta^\pm, \quad \mathrm{ev}_{\mathbf{a}}^\pm(F_0) = (-q)^{n-1} \mathbf{a}^{-1} K_{\mp(\Lambda_1 - \Lambda_n)}^{'} E_\theta^\pm.$$

From Remark 4.5 and Proposition 4.6, we can regard any representation of U_q as a representation of $\widetilde{U}_q$.

Definition 4.7. For $\lambda \in P_+$ and $\mathbf{a} \in \mathbb{C}(q)^\times$, we set

$$\mathbf{a}_\lambda^+ := \mathbf{a} q^{-c_{1,\lambda} + c_{n,\lambda} + n}, \quad \mathbf{a}_\lambda^- := (-1)^{n+1} \mathbf{a} q^{c_{1,\lambda} - c_{n,\lambda} + 2n+1}.$$

We regard $V_q(\lambda)$ as a representation of $\widetilde{U}_q$ by using $\mathrm{ev}_{\mathbf{a}_\lambda^\pm}^\pm$ and denote them by $V_q(\lambda)_{\mathbf{a}}^\pm$ which are called *evaluation representations* of $V_q(\lambda)$.

For $i \in I$, $\lambda \in P_+$, and $\mathbf{a} \in \mathbb{C}(q)^\times$, we define $\pi_i^{\mathbf{a},\lambda} = (\pi_{i,j}^{\mathbf{a},\lambda}(t))_{j \in I} \in (\mathbb{C}(q)_0[t])^n$ whereby

$$\pi_{i,j}^{\mathbf{a},\lambda}(t) := \begin{cases} \prod_{k=1}^{\lambda} (1 - \mathbf{a} q^{\lambda-2k+1} t), & \text{if} \quad j = i, \\ 1, & \text{if} \quad j \neq i, \end{cases} \tag{4.10}$$

where $\prod_{k=1}^{\lambda}(1 - \mathbf{a} q^{\lambda-2k+1} t) := 1$ if $\lambda = 0$. For $\pi = (\pi_i(t))_{i \in I}$, $\pi^{'} = (\pi_i^{'}(t))_{i \in I} \in (\mathbb{C}(q)_0[t])^n$, we define $\pi\pi^{'} := (\pi_i(t)\pi_i^{'}(t))_{i \in I}$. For $i \in I$ and $\lambda = \sum_{i \in I} \lambda_i \Lambda_i \in P_+$, we define

$$\lambda[i] := -\sum_{k=1}^{i-1} \lambda_k + \sum_{k=i+1}^{n} \lambda_k - i. \tag{4.11}$$

The following theorem was proved by Chari and Pressley in [CP94a] (see also [AN], IV).

Theorem 4.8 ([CP94a]). *For $\lambda = \sum_{i\in I} \lambda_i\Lambda_i \in P_+$ and $\mathbf{a} \in \mathbb{C}(q)^\times$, as representations of $\widetilde{U}_q$,*

$$V_q(\lambda)^{\pm}_{\mathbf{a}} \cong \widetilde{V}_q(\prod_{i\in I} \pi_i^{\mathbf{a}q^{\pm\lambda[i]},\lambda_i}).$$

In particular, we have

$$H_{i,1}v_\lambda = \mathbf{a}q^{\pm\lambda[i]-1}[\lambda_i]_q v_\lambda, \tag{4.12}$$

where v_λ is a highest-weight vector in $V_q(\lambda)$.

4.4 The fundamental representations of $\widetilde{U}_q$

For $\xi \in I$ and $\mathbf{a} \in \mathbb{C}(q)^\times$, we set $\pi^{\mathbf{a}}_\xi := \pi^{\mathbf{a},1}_\xi$. Hence $\pi^{\mathbf{a}}_\xi = (\pi^{\mathbf{a}}_{\xi,j}(t))_{j\in I}$ is an element in $(\mathbb{C}(q)_0[t])^n$ such that

$$\pi^{\mathbf{a}}_{\xi,j}(t) = \begin{cases} 1-\mathbf{a}t, & \text{if} \quad j=\xi, \\ 1, & \text{if} \quad j\neq\xi. \end{cases} \tag{4.13}$$

We call $\widetilde{V}_q(\pi^{\mathbf{a}}_\xi)$ a *fundamental representation* of $\widetilde{U}_q$.

For $\lambda \in P_+$ and $\mathbf{a} \in \mathbb{C}(q)^\times$, let $V_q(\lambda)^{\pm}_{\mathbf{a}}$ be as in Definition 4.7. For $\xi \in I$, we set

$$V_q(\Lambda_\xi)_{\mathbf{a}} := V_q(\Lambda_\xi)^{+}_{\mathbf{a}q^\xi}. \tag{4.14}$$

From Theorem 4.8, we have

$$V_q(\Lambda_\xi)_{\mathbf{a}} = V_q(\Lambda_\xi)^{+}_{\mathbf{a}q^\xi} \cong V_q(\Lambda_\xi)^{-}_{\mathbf{a}q^{-\xi}} \cong \widetilde{V}_q(\pi^{\mathbf{a}}_\xi) \quad \text{as representations of } \widetilde{U}_q. \tag{4.15}$$

So we can regard $V_q(\Lambda_\xi)_{\mathbf{a}}$ as the fundamental representation of $\widetilde{U}_q$. From Proposition 3.11, we obtain the following proposition.

Proposition 4.9. *Let $\xi \in I$ and $\mathbf{a} \in \mathbb{C}(q)^\times$.*

(a) There exists an integer $c \in \mathbb{Z}$ depending only on $\mathfrak{sl}_{n+1}$ such that $V_q(\Lambda_\xi)^*_{\mathbf{a}}$ is isomorphic to $V_q(\Lambda_{n-\xi+1})_{q^c\mathbf{a}}$ as a representation of $\widetilde{U}_q$.

(b) There exists a nonzero complex number $\kappa \in \mathbb{C}^\times$ depending only on $\mathfrak{sl}_{n+1}$ such that $V_q(\Lambda_\xi)^{\Omega}_{\mathbf{a}}$ is isomorphic to $V_q(\Lambda_{n-\xi+1})_{q^2\kappa\mathbf{a}^{-1}}$ as a representation of $\widetilde{U}_q$. In particular, $z_{\omega_{1,n}\Lambda_\xi}$ is a pseudo-highest weight vector in $V_q(\Lambda_\xi)^{\Omega}_{\mathbf{a}}$.

From (4.12), for $i \in I$, we have

$$H_{i,1}z_{\Lambda_\xi} = \Lambda_{\pi^{\mathbf{a}}_\xi}(H_{i,1})z_{\Lambda_\xi} = \mathbf{a}q^{-1}\delta_{i,\xi}z_{\Lambda_\xi} \quad \text{in} \quad V_q(\Lambda_\xi)_{\mathbf{a}}. \tag{4.16}$$

Proposition 4.10. *Let $\xi \in I$ and $\mathbf{a} \in \mathbb{C}(q)^\times$. For $i, j \in I$ such that $i \leq j$, let $\omega_{i,j}$ be as in (3.18) and let $z_{\Lambda_\xi}(\omega_{i,j})$ be the extremal vector in $V_q(\Lambda_\xi)_{\mathbf{a}}$.*

(a) We have $z_{\Lambda_\xi}(\omega_{i,j}) = z_{\omega_{i,j}\Lambda_\xi}$.

(b) We have

$$\begin{aligned} &H_{i-1,1}z_{\Lambda_\xi}(\omega_{i,j}) = \mathbf{a}q^{2j-i-\xi}\delta(j-i+2 \leq \xi \leq j)z_{\Lambda_\xi}(\omega_{i,j}), \quad \textit{if } i \neq 1, \\ &H_{j+1,1}z_{\Lambda_\xi}(\omega_{1,j}) = \mathbf{a}q^{j-\xi}\delta(\xi \leq j+1)z_{\Lambda_\xi}(\omega_{1,j}). \end{aligned}$$

Proof. (b) follows from Proposition 3.13 and (4.16). So we shall prove (a). We obtain

$$z_{\Lambda_\xi}(\omega_{1,1}) = F_1^{\delta_{\xi,1}}z_{\Lambda_\xi} = \delta_{\xi,1}z_{\Lambda_\xi-\alpha_1} + (1-\delta_{\xi,1})z_{\Lambda_\xi} = \delta_{\xi,1}z_{s_1\Lambda_\xi} + (1-\delta_{\xi,1})z_{s_1\Lambda_\xi} = z_{\omega_{1,1}\Lambda_\xi}.$$

Now we assume $z_{\Lambda_\xi}(\omega_{i,j}) = z_{\omega_{i,j}\Lambda_\xi}$. From (3.21), if $i \neq 1$, we obtain

$$z_{\Lambda_\xi}(\omega_{i-1,j}) = F_{i-1}^{\delta(j-i+2\leq\xi\leq j)}z_{\Lambda_\xi}(\omega_{i,j}) = F_{i-1}^{\delta(j-i+2\leq\xi\leq j)}z_{\omega_{i,j}\Lambda_\xi}.$$

Thus, from (3.19) and (4.3), we have

$$\begin{aligned} z_{\Lambda_\xi}(\omega_{i-1,j}) &= \delta(j-i+2\le\xi\le j) z_{\omega_{i,j}\Lambda_\xi-\alpha_{i-1}} + (1-\delta(j-i+2\le\xi\le j)) z_{\omega_{i,j}\Lambda_\xi} \\ &= z_{s_{i-1}\omega_{i,j}\Lambda_\xi} = z_{\omega_{i-1,j}\Lambda_\xi}. \end{aligned}$$

Similarly, if $i=1$, by using (3.20) and (3.22), we obtain $z_{\Lambda_\xi}(\omega_{1,j}) = z_{\omega_{1,j}\Lambda_\xi}$. $\square$

Let $L_q(\Lambda_\xi)$ and J_H^ξ be as in §4.2. From Proposition 4.3, Proposition 4.6, and Definition 4.7, we have

$$E_0 J_H^\xi = \mathbf{a} q^{n+1}(-1)^{\xi-1}\{2,3,\cdots,\xi,n+1\} \quad \text{in} \quad L_q(\Lambda_\xi)_{\mathbf{a}} \,(\cong V_q(\Lambda_\xi)_{\mathbf{a}}). \tag{4.17}$$

Moreover, for $J \in \mathcal{J}_\xi$, we have

$$\begin{aligned} E_0 J &= \mathbf{a} q^{n-1}(-1)^\xi((J\backslash\{1\}) \sqcup \{n+1\}), \\ F_0 J &= \mathbf{a}^{-1} q^{-n+1}(-1)^\xi((J\backslash\{n+1\}) \sqcup \{1\}), \end{aligned}$$

(see [DO] and [AK]).

Proposition 4.11. *For $i,j\in I$ such that $i\le j$, let $J_H^\xi(\omega_{i,j})$ be the extremal vector in $L_q(\Lambda_\xi)_{\mathbf{a}}$. We have*

$$J_H^\xi(\omega_{i,j}) = \begin{cases} J_H^\xi, & \text{if } j<\xi, \\ \{j+2-\xi, j+3-\xi,\cdots,j+1\}, & \text{if } \xi\le j \text{ and } i\le j+1-\xi, \\ \{j+1-\xi,\cdots,i-1,i+1,\cdots,j+1\}, & \text{if } \xi\le j \text{ and } j+1-\xi<i. \end{cases}$$

Proof. This proposition follows from Proposition 4.3 and the definition of the extremal vectors in §3.4. $\square$

From Lemma 4.2 in [C], we obtain the following lemma.

Lemma 4.12. *Let $\xi\in I$, $\mathbf{a}\in\mathbb{C}(q)^\times$, and $\pi\in(\mathbb{C}(q)_0[t])^n$. Let V be a pseudo-highest weight representation of $\widetilde{U}_q$ with the pseudo-highest weight π generated by a pseudo-highest weight vector v_π. If $z_{\omega_{1,n}\Lambda_\xi}\otimes v_\pi \in \widetilde{U}_q(z_{\Lambda_\xi}\otimes v_\pi)$, then $V_q(\Lambda_\xi)_{\mathbf{a}}\otimes V$ is a pseudo-highest weight representation of $\widetilde{U}_q$ with the pseudo-highest weight $\pi_\xi^{\mathbf{a}}\pi$ generated by a pseudo-highest weight vector $z_{\Lambda_\xi}\otimes v_\pi$.*

Proof. From Lemma 3.8, it is enough to prove that

$$V_q(\Lambda_\xi)_{\mathbf{a}}\otimes V = \widetilde{U}_q(z_{\Lambda_\xi}\otimes v_\pi).$$

Moreover, from the assumption of this Lemma, it is enough to prove that

$$V_q(\Lambda_\xi)_{\mathbf{a}}\otimes V \subset \widetilde{U}_q(z_{\omega_{1,n}\Lambda_\xi}\otimes v_\pi).$$

Since v_π is a pseudo-highest weight vector, for any $i_1,i_2,\cdots,i_r\in I$, we have

$$\begin{aligned} E_{i_1}E_{i_2}\cdots E_{i_r}(z_{\omega_{1,n}\Lambda_\xi}\otimes v_\pi) &= (E_{i_1}E_{i_2}\cdots E_{i_r} z_{\omega_{1,n}\Lambda_\xi})\otimes v_\pi \quad \text{in} \quad V_q(\Lambda_\xi)_{\mathbf{a}}\otimes V \\ &= (F_{i_1}F_{i_2}\cdots F_{i_r} z_{\omega_{1,n}\Lambda_\xi})\otimes v_\pi \quad \text{in} \quad V_q(\Lambda_\xi)_{\mathbf{a}}^\Omega\otimes V. \end{aligned}$$

Hence, from Proposition 4.9 (b), we obtain

$$\widetilde{U}_q(z_{\omega_{1,n}\Lambda_\xi}\otimes v_\pi) \supset (U_q^- z_{\omega_{1,n}\Lambda_\xi})\otimes v_\pi = V_q(\Lambda_\xi)_{\mathbf{a}}\otimes v_\pi.$$

Let $Y_{j_1},Y_{j_2},\cdots,Y_{j_s}\in\{E_i,F_i\,|\,i\in\widetilde{I}\}$. We assume $V_q(\Lambda_\xi)_{\mathbf{a}}\otimes(Y_{j_1}\cdots Y_{j_s}v_\pi)\subset\widetilde{U}_q(z_{\omega_{1,n}\Lambda_\xi}\otimes v_\pi)$. Then, for any $\mu\in\mathcal{W}\Lambda_\xi$, we have

$$\begin{aligned} q^{(\mu,\alpha_i)}(z_\mu\otimes E_i(Y_{j_1}\cdots Y_{j_s}v_\pi)) &= E_i(z_\mu\otimes(Y_{j_1}\cdots Y_{j_s}v_\pi)) - (E_i z_\mu)\otimes(Y_{j_1}\cdots Y_{j_s}v_\pi) \in \widetilde{U}_q(z_{\omega_{1,n}\Lambda_\xi}\otimes v_\pi), \\ z_\mu\otimes F_i(Y_{j_1}\cdots Y_{j_s}v_\pi) &= F_i(z_\mu\otimes(Y_{j_1}\cdots Y_{j_s}v_\pi)) - q^{-(\mu,\alpha_i)}(F_i z_\mu)\otimes(Y_{j_1}\cdots Y_{j_s}v_\pi) \in \widetilde{U}_q(z_{\omega_{1,n}\Lambda_\xi}\otimes v_\pi). \end{aligned}$$

Therefore we obtain

$$\widetilde{U}_q(z_{\omega_{1,n}\Lambda_\xi}\otimes v_\pi) \supset V_q(\Lambda_\xi)_{\mathbf{a}}\otimes\widetilde{U}_q v_\pi = V_q(\Lambda_\xi)_{\mathbf{a}}\otimes V.$$

$\square$

4.5 The fundamental representations of $U_q(\widetilde{\mathfrak{sl}}_2)$

We denote the generators $X^{\pm}_{1,r}, H_{1,s}, K_1^{\pm 1}$ in $U_q(\widetilde{\mathfrak{sl}}_2)$ (resp. $E_1, F_1, K_1^{\pm 1}$ in $U_q(\mathfrak{sl}_2)$) by $X^{\pm}_r$, H_s, $K^{\pm 1}$ (resp. E, F, $K^{\pm 1}$) and the fundamental weight Λ_1 by Λ. For $\mathbf{a} \in \mathbb{C}(q)^{\times}$, we denote the $U_q(\widetilde{\mathfrak{sl}}_2)$-representation $\widetilde{V}_q(\pi_1^{\mathbf{a}})$ (resp. the $U_q(\mathfrak{sl}_2)$-representation $V_q(\Lambda)$) by $\widetilde{V}_q(\pi^{\mathbf{a}})$ (resp. $V_q(1)$). Then $V_q(1)_{\mathbf{a}}$ ($\cong \widetilde{V}_q(\pi^{\mathbf{a}})$) has the following realization:

$$V_q(1)_{\mathbf{a}} = \mathbb{C}(q) z_{\Lambda} \oplus \mathbb{C}(q) z_{-\Lambda}, \quad X^{-}_r z_{\Lambda} = \mathbf{a}^r z_{-\Lambda}, \quad X^{+}_r z_{-\Lambda} = \mathbf{a}^r z_{\Lambda}, \quad X^{+}_r z_{\Lambda} = X^{-}_r z_{-\Lambda} = 0, \tag{4.18}$$

for any $r \in \mathbb{Z}$. Moreover, for $m \in \mathbb{N}$, we have

$$\Psi^{+}_m z_{\Lambda} = \mathbf{a}^m (q - q^{-1}) z_{\Lambda}, \quad \Psi^{+}_0 z_{\Lambda} = K z_{\Lambda} = q z_{\Lambda}. \tag{4.19}$$

5 Tensor product of the fundamental representations for the quantum loop algebras: the generic case

5.1 Irreducibility: the $\widetilde{U}_q$ case

In this subsection, we review the results and proofs in [C] that will be needed later.

For $i \in I$, let $\widetilde{U}_q^{(i)}$ (resp. $U_q^{(i)}$) be the $\mathbb{C}(q)$-subalgebra of $\widetilde{U}_q$ (resp. U_q) generated by $\{X^{\pm}_{i,r}, H_{i,s}, K_i^{\pm 1} \mid r \in \mathbb{Z}, s \in \mathbb{Z}^{\times}\}$ (resp. $\{E_i, F_i, K_i^{\pm 1}\}$). There exist $\mathbb{C}(q)$-algebra homomorphisms $\widetilde{\iota} : U_q(\widetilde{\mathfrak{sl}}_2) \longrightarrow \widetilde{U}_q^{(i)}$ and $\iota : U_q(\mathfrak{sl}_2) \longrightarrow U_q^{(i)}$ such that

$$\begin{aligned} &\widetilde{\iota}(X^{\pm}_r) = X^{\pm}_{i,r}, \quad \widetilde{\iota}(H_s) = H_{i,s}, \quad \widetilde{\iota}(K^{\pm 1}) = K_i^{\pm 1}, \\ &\iota(E) = E_i, \quad \iota(F) = F_i, \quad \iota(K^{\pm 1}) = K_i^{\pm 1}, \end{aligned}$$

for any $i \in I$, $r \in \mathbb{Z}$, and $s \in \mathbb{Z}^{\times}$. Hence, for any $\widetilde{U}_q^{(i)}$-representation (resp. $U_q^{(i)}$-representation) V, we can regard V as a $U_q(\widetilde{\mathfrak{sl}}_2)$-representation (resp. $U_q(\mathfrak{sl}_2)$-representation).

Lemma 5.1. *Let $\xi \in I$ and $\mathbf{a} \in \mathbb{C}(q)^{\times}$. For any $i, j \in I$ such that $i \leq j$, as representations of $U_q(\widetilde{\mathfrak{sl}}_2)$,*

$$\widetilde{U}_q^{(i-1)} z_{\omega_{i,j}\Lambda_{\xi}} \cong \begin{cases} V_q(1)_{\mathbf{a}q^{2j-\xi-i+1}}, & \text{if} \quad j-i+2 \leq \xi \leq j, \\ V_q(0)_{\mathbf{a}}, & \text{otherwise}, \end{cases} \qquad \text{if} \quad i \neq 1,$$

$$\widetilde{U}_q^{(j+1)} z_{\omega_{1,j}\Lambda_{\xi}} \cong \begin{cases} V_q(1)_{\mathbf{a}q^{j-\xi+1}}, & \text{if} \quad 1 \leq \xi \leq j+1, \\ V_q(0)_{\mathbf{a}}, & \text{otherwise}. \end{cases}$$

Proof. We assume $i \neq 1$. We can prove the case of $i = 1$ similarly. We set $\mu := \omega_{i,j}\Lambda_{\xi}$. From (3.17), for any $r \in \mathbb{Z}$, $X^{+}_{i-1,r} z_{\mu} = 0$. Hence we have

$$\begin{aligned} \widetilde{U}_q^{(i-1)} z_{\mu} \quad &\subset \quad \Big(\sum_{r_1, \cdots, r_m \in \mathbb{Z}, m \in \mathbb{N}} \mathbb{C}(q) X^{-}_{i-1,r_1} \cdots X^{-}_{i-1,r_m} z_{\mu} \Big) \oplus \mathbb{C}(q) z_{\mu} \\ &\subset \quad \bigoplus_{m \in \mathbb{N}} (V_q(\Lambda_{\xi})_{\mu - m\alpha_{i-1}}) \oplus \mathbb{C}(q) z_{\mu}. \end{aligned}$$

Since Λ_{ξ} is a minuscule weight (see §4.1), for any $\nu \in \mathcal{W}\Lambda_{\xi}$, we have $|\langle \nu, \alpha^{\vee}_{i-1} \rangle| \leq 1$. Thus we obtain

$$|\langle \mu - m\alpha_{i-1}, \alpha^{\vee}_{i-1} \rangle| = |\langle \mu, \alpha^{\vee}_{i-1} \rangle - 2m| \geq 2m - 1 \quad \text{for} \quad m \in \mathbb{N}.$$

So if $m \geq 2$, $V_q(\Lambda_{\xi})_{\mu - m\alpha_i} = 0$. Hence we obtain

$$\widetilde{U}_q^{(i-1)} z_{\mu} \subset V_q(\Lambda_{\xi})_{\mu - \alpha_{i-1}} \oplus \mathbb{C}(q) z_{\mu}.$$

Since $\dim_{\mathbb{C}(q)}(V_q(\Lambda_\xi)_{\mu-\alpha_{i-1}}) \le 1$ from (4.4), we have $\dim_{\mathbb{C}(q)}(\widetilde{U}_q^{(i-1)} z_\mu) \le 2$. So $\widetilde{U}_q^{(i-1)} z_\mu \cong V_q(0)_{\mathbf{a}}$ or there exists an element $\mathbf{b} \in \mathbb{C}(q)^\times$ such that $\widetilde{U}_q^{(i-1)} z_\mu \cong V_q(1)_{\mathbf{b}}$ as representations of $U_q(\widetilde{\mathfrak{sl}}_2)$. From (3.19), we have

$$\langle \mu, \alpha_{i-1}^\vee \rangle = \langle \omega_{i,j}\Lambda_\xi, \alpha_{i-1}^\vee \rangle = \delta(j-i+2 \le \xi \le j).$$

Therefore, from (4.3), we obtain

$$\widetilde{U}_q^{(i-1)} z_\mu \cong \begin{cases} V_q(1)_{\mathbf{b}}, & \text{if} \quad j-i+2 \le \xi \le j, \\ V_q(0)_{\mathbf{a}}, & \textit{otherwise}. \end{cases}$$

We assume $j-i+2 \le \xi \le j$. From (4.16), we have

$$H_{i-1,1} z_\mu = \mathbf{b} q^{-1} z_\mu \quad \text{in} \quad V_q(1)_{\mathbf{b}}.$$

On the other hand, from Proposition 4.10 (b), we have

$$H_{i-1,1} z_\mu = \mathbf{a} q^{2j-i-\xi} z_\mu \quad \text{in} \quad V_q(\Lambda_\xi)_{\mathbf{a}}.$$

Therefore we obtain $\mathbf{b} = \mathbf{a} q^{2j-i-\xi+1}$. □

Let $m \in \mathbb{N}$ ($m \ge 2$) and $\mathbf{a} = (\mathbf{a}_1, \cdots, \mathbf{a}_m) \in (\mathbb{C}(q)^\times)^m$. For $1 \le r, s \le m$, we define

$$A^q_{r,s}(\mathbf{a}) := q^{m-s}\mathbf{a}_s^r + \sum_{k=1}^{r-1} \mathbf{a}_s^{r-k} d^q_{k,s}(\mathbf{a}), \quad A^q(\mathbf{a}) := (A^q_{r,s}(\mathbf{a}))_{r,s=1}^m, \tag{5.1}$$

where, around $u = 0$,

$$\sum_{k=0}^{\infty} d^q_{k,s}(\mathbf{a}) u^k := q^m \frac{(1-q^{-2}\mathbf{a}_{s+1}u)\cdots(1-q^{-2}\mathbf{a}_m u)}{(1-\mathbf{a}_{s+1}u)\cdots(1-\mathbf{a}_m u)} \quad \text{in} \quad \mathbb{C}(q)[[u]], \tag{5.2}$$

(see (3.10)) and $d^q_{k,m}(\mathbf{a}) := 0$ for any $k \in \mathbb{Z}_+$. So we have

$$\sum_{k=0}^{\infty} d^q_{k,s}(\mathbf{a}) u^k = \prod_{p=s+1}^{m} \{q + (q-q^{-1})(\mathbf{a}_p u + \mathbf{a}_p^2 u^2 + \mathbf{a}_p^3 u^3 + \cdots)\}$$

From the proof of Lemma 4.10 in [CP91], we obtain the following lemma.

Lemma 5.2 ([CP91], §4.10)**.**

$$\det(A^q(\mathbf{a})) = (\prod_{k=1}^{m} \mathbf{a}_k)(\prod_{1 \le s < t \le m} (q^{-1}\mathbf{a}_t - q\mathbf{a}_s)). \tag{5.3}$$

The following theorem is the special case of Theorem 4.4 in [C].

Theorem 5.3 ([C], Theorem 4.4)**.** *Let* $m \in \mathbb{N}$, $\xi_1, \cdots, \xi_m \in I$, *and* $\mathbf{a}_1, \cdots, \mathbf{a}_m \in \mathbb{C}^\times$. *We assume that for any* $1 \le k < k^{'} \le m$ *and* $\max(\xi_k, \xi_{k'}) \le t \le \min(\xi_k + \xi_{k'} - 1, n)$,

$$\frac{\mathbf{a}_{k'}}{\mathbf{a}_k} \ne q^{2t-\xi_k-\xi_{k'}+2}.$$

Then $V_q(\Lambda_{\xi_1})_{\mathbf{a}_1} \otimes \cdots \otimes V_q(\Lambda_{\xi_m})_{\mathbf{a}_m}$ *is a pseudo-highest weight representation of* $\widetilde{U}_q$ *with the pseudo-highest weight* $\pi_{\xi_1}^{\mathbf{a}_1} \cdots \pi_{\xi_m}^{\mathbf{a}_m}$ *generated by a pseudo-highest weight vector* $z_{\Lambda_{\xi_1}} \otimes \cdots \otimes z_{\Lambda_{\xi_m}}$.

Proof. We prove this theorem by using the method in [C] and [CP91]. From Proposition 3.8, it is enough to prove that

$$V_q(\Lambda_{\xi_1})_{\mathbf{a}_1} \otimes \cdots \otimes V_q(\Lambda_{\xi_m})_{\mathbf{a}_m} = \widetilde{U}_q(z_{\Lambda_{\xi_1}} \otimes \cdots \otimes z_{\Lambda_{\xi_m}}).$$

We shall prove this claim by the induction on m. If $m = 1$, we have nothing to prove. So we assume $m > 1$ and the case of $(m-1)$ holds. We set

$$V^{'} := V_q(\Lambda_{\xi_2})_{\mathbf{a}_2} \otimes \cdots \otimes V_q(\Lambda_{\xi_m})_{\mathbf{a}_m}, \quad z^{'} := (z_{\Lambda_{\xi_2}} \otimes \cdots \otimes z_{\Lambda_{\xi_m}}).$$

From Proposition 3.8 and the assumption of the induction on m, $V^{'}$ is a pseudo-highest weight representation of $\widetilde{U}_q$ with the pseudo-highest weight $\pi_{\xi_2}^{\mathbf{a}_2} \cdots \pi_{\xi_m}^{\mathbf{a}_m}$ generated by a pseudo-highest weight vector $z^{'}$. Hence, from Lemma 4.12, it is enough to prove that

$$z_{\omega_{1,n}\Lambda_{\xi_1}} \otimes z^{'} \in \widetilde{U}_q(z_{\Lambda_{\xi_1}} \otimes z^{'}).$$

We shall prove that

$$z_{\omega_{i,j}\Lambda_{\xi_1}} \otimes z^{'} \in \widetilde{U}_q(z_{\Lambda_{\xi_1}} \otimes z^{'}), \tag{5.4}$$

for any $i, j \in I$ such that $i \leq j$. We set $z_{\omega_{1,0}\Lambda_{\xi_1}} := z_{\Lambda_{\xi_1}}$. We define a total order in $I^{\leq} := \{(i,j) \,|\, 1 \leq i \leq j \leq n\} \sqcup \{(1,0)\}$ whereby

$$(i-1, j) > (i, j), \quad \text{and} \quad (j+1, j+1) > (1, j), \tag{5.5}$$

for $2 \leq i \leq n$ and $0 \leq j \leq n$. We shall prove (5.4) by the induction on (i,j). If $(i,j) = (1,0)$, we have nothing to prove. So we assume $(i,j) \neq (1,0)$ and the case of (i,j) holds. We also assume $i \neq 1$. We can prove the case of $i = 1$ similarly. From (3.19), we have $\langle \omega_{i,j}, \alpha_{i-1}^{\vee} \rangle = \delta(j - i + 2 \leq \xi \leq j)$. So if $\xi_1 < j - i + 2$ or $\xi_1 > j$, then $\omega_{i-1,j} = s_{i-1}\omega_{i,j} = \omega_{i,j}$. Hence

$$z_{\omega_{i-1,j}\Lambda_{\xi_1}} \otimes z^{'} = z_{\omega_{i,j}\Lambda_{\xi_1}} \otimes z^{'} \in \widetilde{U}_q(z_{\Lambda_{\xi_1}} \otimes z^{'}),$$

by the induction on (i,j). So we assume $j - i + 2 \leq \xi_1 \leq j$.

Case 1) In the case of $\xi_2 = \cdots = \xi_m = i - 1$: From Proposition 3.4 (b), for $r \in \mathbb{Z}$, we have

$$\begin{aligned} &\Delta_H(X_{i-1,r}^-)(z_{\omega_{i,j}\Lambda_{\xi_1}} \otimes z^{'}) - z_{\omega_{i,j}\Lambda_{\xi_1}} \otimes (\Delta_H(X_{i-1,r}^-)z^{'}) \\ &= (\Delta_H(X_{i-1,r-k}^-)z_{\omega_{i,j}\Lambda_{\xi_1}}) \otimes (\Delta_H(K_{i-1})z^{'}) + \sum_{k=1}^{r-1}(X_{i-1,r-k}^- z_{\omega_{i,j}\Lambda_{\xi_1}}) \otimes (\Delta_H(\Psi_{i-1,k}^+)z^{'}). \end{aligned} \tag{5.6}$$

We have

$$\Delta_H(K_{i-1})z^{'} = q^{m-1}z^{'}. \tag{5.7}$$

We set

$$\widetilde{\mathbf{a}}_1 := \mathbf{a}_1 q^{2j-\xi_1-i+1}, \quad \widetilde{\mathbf{a}}_2 := \mathbf{a}_2, \quad \cdots, \quad \widetilde{\mathbf{a}}_m := \mathbf{a}_m, \quad \widetilde{\mathbf{a}} := (\widetilde{\mathbf{a}}_1, \cdots, \widetilde{\mathbf{a}}_m).$$

From Lemma 5.1, $\widetilde{U}_q^{(i)} z_{\omega_{i,j}\Lambda_{\xi_1}} \cong V_q(1)_{\widetilde{\mathbf{a}}_1}$ as representations of $U_q(\widetilde{\mathfrak{sl}}_2)$ and $z_{\omega_{i,j}\Lambda_{\xi_1}}$ is a pseudo-highest weight vector in $\widetilde{U}_q^{(i)} z_{\omega_{i,j}\Lambda_{\xi_1}}$. Hence, from (4.18), we have

$$\Delta_H(X_{i-1,r}^-)z_{\omega_{i,j}\Lambda_{\xi_1}} = \widetilde{\mathbf{a}}_1^r F_{i-1} z_{\omega_{i,j}\Lambda_{\xi_1}} = \widetilde{\mathbf{a}}_1^r z_{\omega_{i-1,j}\Lambda_{\xi_1}}. \tag{5.8}$$

From (3.10), for $1 \leq k \leq r-1$, we obtain

$$\Delta_H(\Psi_{i-1,k})z^{'} = d_{k,1}^q(\widetilde{\mathbf{a}})z^{'}, \tag{5.9}$$

where $d^q_{k,1}(\widetilde{\mathbf{a}})$ be as in (5.2). From (5.6)–(5.9), we obtain

$$\begin{aligned}
&\Delta_H(X^-_{i-1,r})(z_{\omega_{i,j}\Lambda_{\xi_1}} \otimes z^{'}) - z_{\omega_{i,j}\Lambda_{\xi_1}} \otimes (\Delta_H(X^-_{i-1,r})z^{'}) \\
&= (q^{m-1}\widetilde{\mathbf{a}}_1^r + \sum_{k=1}^{r-1} \widetilde{\mathbf{a}}_1^{r-k} d^q_{k,1}(\widetilde{\mathbf{a}}))(z_{\omega_{i-1,j}\Lambda_{\xi_1}} \otimes z^{'}) = A^q_{r,1}(\widetilde{\mathbf{a}})(z_{\omega_{i-1,j}\Lambda_{\xi_1}} \otimes z^{'}).
\end{aligned}$$

By repeating this procedure m-times, we obtain

$$\begin{aligned}
&\Delta_H(X^-_{i-1,r})(z_{\omega_{i,j}\Lambda_{\xi_1}} \otimes z^{'}) \\
&= A^q_{r,1}(\widetilde{\mathbf{a}})(z_{\omega_{i-1,j}\Lambda_{\xi_1}} \otimes z^{'}) + \sum_{s=2}^{m} A^q_{r,s}(\widetilde{\mathbf{a}})(z_{\omega_{i,j}\Lambda_{\xi_1}} \otimes z_{\Lambda_{\xi_2}} \otimes \cdots \otimes (F_{i-1}z_{\Lambda_{\xi_s}}) \otimes \cdots \otimes z_{\Lambda_{\xi_m}}).
\end{aligned}$$

Since $\Delta_H(X^-_{i-1,r})(z_{\omega_{i,j}\Lambda_{\xi_1}} \otimes z^{'}) \in \widetilde{U}_q(z_{\Lambda_{\xi_1}} \otimes z^{'})$ from the assumption of the induction on (i,j), we obtain

$$\det(A^q(\widetilde{\mathbf{a}}))(z_{\omega_{i-1,j}\Lambda_{\xi_1}} \otimes z^{'}) \in \widetilde{U}_q(z_{\Lambda_{\xi_1}} \otimes z^{'}),$$

where $A^q(\widetilde{\mathbf{a}}) = (A^q_{r,s}(\widetilde{\mathbf{a}}))^m_{r,s=1}$ be as in (5.1). Hence, from Lemma 5.2 and the assumption of Case 1, we have

$$(\prod_{k=2}^{m}(\mathbf{a}_k - \mathbf{a}_1 q^{2j-\xi_1-\xi_k+2}))(\prod_{2\le k<k^{'}\le m}(\mathbf{a}_{k^{'}} - \mathbf{a}_k q^2))(z_{\omega_{i-1,j}\Lambda_{\xi_1}} \otimes z^{'}) \in \widetilde{U}_q(z_{\Lambda_{\xi_1}} \otimes z^{'}). \tag{5.10}$$

From the assumption of this theorem, for any $1 \le k \le m$, we have

$$\mathbf{a}_k - \mathbf{a}_1 q^{2j-\xi_1-\xi_k+2} \neq 0. \tag{5.11}$$

Moreover, if $\xi_k = \xi_{k^{'}}$, we have $\max(\xi_k, \xi_{k^{'}}) \le \xi_k = \xi^{'}_k \le \min(\xi_k + \xi_{k^{'}} - 1, n)$. Thus, from the assumption of this theorem and Case 1, we have

$$\mathbf{a}_{k^{'}} - \mathbf{a}_k q^2 \neq 0. \tag{5.12}$$

for any $2 \le k < k^{'} \le m$. Therefore, from (5.10)–(5.12), we obtain

$$z_{\omega_{i-1,j}\Lambda_{\xi_1}} \otimes z^{'} \in \widetilde{U}_q(z_{\Lambda_{\xi_1}} \otimes z^{'}).$$

Case 2) There exists an integer $m^{'}$ $(2 \le m^{'} \le m)$ such that $\xi_{m^{'}} \neq i-1$: We set

$$M := \{2 \le m^{'} \le m \,|\, \xi_{m^{'}} = i-1\}. \tag{5.13}$$

If $M = \emptyset$, then $F_{i-1}z^{'} = 0$. Hence we obtain

$$\widetilde{U}_q(z_{\Lambda_{\xi_1}} \otimes z^{'}) \ni F_{i-1}(z_{\omega_{i,j}\Lambda_{\xi_1}} \otimes z^{'}) = q^{-1}(z_{\omega_{i-1,j}\Lambda_{\xi_1}} \otimes z^{'}).$$

We assume $M \neq \emptyset$. For $m^{'} \in \{1, 2, \cdots, m\}$ such that $\xi_{m^{'}} \neq i-1$, we have

$$\Delta_H(X^-_{i-1,r})(z_{\xi_{m^{'}}} \otimes z_{\xi_{m^{'}+1}} \otimes \cdots \otimes z_{\xi_m}) = z_{\xi_{m^{'}}} \otimes \Delta_H(X^-_{i-1,r})(z_{\xi_{m^{'}+1}} \otimes \cdots \otimes z_{\xi_m}),$$

(see (5.6)). Hence, in a similar way to the proof of Case 1, we obtain

$$(\prod_{k\in M}(\mathbf{a}_k - \mathbf{a}_1 q^{2j-\xi_1-\xi_k+2}))(\prod_{k,k^{'}\in M, k<k^{'}}(\mathbf{a}_{k^{'}} - \mathbf{a}_k q^2))(z_{\omega_{i-1,j}\Lambda_{\xi_1}} \otimes z^{'}) \in \widetilde{U}_q(z_{\Lambda_{\xi_1}} \otimes z^{'}).$$

Therefore, from the assumption of this theorem, we obtain

$$z_{\omega_{i-1,j}\Lambda_{\xi_1}} \otimes z^{'} \in \widetilde{U}_q(z_{\Lambda_{\xi_1}} \otimes z^{'}).$$

□

Remark 5.4. Since the comultiplication Δ_H in this paper is slightly different from the one in [CP91], $\det(A^q(\mathbf{a}))$ is different from the one in the proof of Lemma 4.10 in [CP91].

Corollary 5.5. *Let $m \in \mathbb{N}$, $\xi_1, \cdots, \xi_m \in I$, and $\mathbf{a}_1, \cdots, \mathbf{a}_m \in \mathbb{C}(q)^\times$. We assume that for any $1 \leq k \neq k^{'} \leq m$ and $\max(\xi_k, \xi_{k'}) \leq t \leq \min(\xi_k + \xi_{k'} - 1, n)$,*

$$\frac{\mathbf{a}_{k'}}{\mathbf{a}_k} \neq q^{\pm(2t-\xi_k-\xi_{k'}+2)}.$$

Then $V_q(\Lambda_{\xi_1})_{\mathbf{a}_1} \otimes \cdots \otimes V_q(\Lambda_{\xi_m})_{\mathbf{a}_m}$ is an irreducible representation of $\widetilde{U}_q$.

Proof. We set $V := V_q(\Lambda_{\xi_1})_{\mathbf{a}_1} \otimes \cdots \otimes V_q(\Lambda_{\xi_m})_{\mathbf{a}_m}$. It is enough to prove that V^Ω is irreducible. Since $(V^\Omega)^\Omega \cong V$, from Theorem 5.3, $(V^\Omega)^\Omega$ is a pseudo-highest representation of $\widetilde{U}_q$. Hence, from Proposition 3.10, it is enough to prove that $(V^\Omega)^*$ is a pseudo-highest representation of $\widetilde{U}_q$. From Proposition 4.9 (b) and (3.15), there exists a nonzero complex number $\kappa \in \mathbb{C}^\times$ such that

$$V^\Omega \cong V_q(\Lambda_{\xi_m})_{q^2\kappa\mathbf{a}_m^{-1}} \otimes \cdots \otimes V_q(\Lambda_{\xi_1})_{q^2\kappa\mathbf{a}_1^{-1}}.$$

Thus, From Proposition 4.9 (a), there exists an integer $c \in \mathbb{Z}$ such that

$$(V^\Omega)^* \cong V_q(\Lambda_{\xi_1})_{q^{c+2}\kappa\mathbf{a}_1^{-1}} \otimes \cdots \otimes V_q(\Lambda_{\xi_m})_{q^{c+2}\kappa\mathbf{a}_m^{-1}}.$$

From the assumption of this corollary, for $1 \leq k < k^{'} \leq m$, we have

$$\frac{q^{c+2}\kappa\mathbf{a}_{k'}^{-1}}{q^{c+2}\kappa\mathbf{a}_k^{-1}} = \frac{\mathbf{a}_k}{\mathbf{a}_{k'}} \neq q^{2t-\xi_k-\xi_{k'}+2}.$$

Therefore, from Theorem 5.3, $(V^\Omega)^*$ is a pseudo-highest representation of $\widetilde{U}_q$. $\square$

5.2 The R-matrices of the fundamental representations of $\widetilde{U}_q$

In this subsection, we regard $V_q(\Lambda_\xi)$ as $L_q(\Lambda_\xi)$ for $\xi \in I$ (see §4.2). We review the R-matrices of the fundamental representations of $\widetilde{U}_q$ introduced in [DO].

For $\xi, \zeta \in I$, as representations of U_q,

$$V_q(\Lambda_\xi) \otimes V_q(\Lambda_\zeta) \cong \bigoplus_{k=\max(0,\xi+\zeta-n-1)}^{\min(\xi,\zeta)} V_q(\Lambda_{\xi+\zeta-k} + \Lambda_k). \tag{5.14}$$

For $k \in I$ such that $\max(0, \xi+\zeta-n-1) \leq k \leq \min(\xi, \zeta)$, we define

$$w_k^{(\xi,\zeta)} := \sum_{J \subset (J^{(\xi+\zeta-k)} \setminus J^{(k)}), \#(J)=\xi-k} (-q)^{\sum_{j\in J} j - \frac{(\xi+k+1)(\xi-k)}{2}} (J^{(k)} \sqcup J) \otimes (J^{(\xi+\zeta-k)} \setminus J), \tag{5.15}$$

where $J^{(i)} := \{1, 2, \cdots, i\}$ $(i \in \widehat{I})$ and $J^{(0)} := \emptyset$.

Let $L_k^{(\xi,\zeta)}$ be the U_q-subrepresentation of $V_q(\Lambda_\xi) \otimes V_q(\Lambda_\zeta)$ such that $L_k^{(\xi,\zeta)} \cong V_q(\Lambda_{\xi+\zeta-k} + \Lambda_k)$. Then $w_k^{(\xi,\zeta)}$ is a highest-weight vector in $L_k^{(\xi,\zeta)}$. For $k \in I$ such that $\max(0, \xi+\zeta-n-1) \leq k \leq \min(\xi, \zeta)$, there exists a U_q-homomorphism $\overline{P}_k : V_q(\Lambda_\xi) \otimes V_q(\Lambda_\zeta) \longrightarrow V_q(\Lambda_\zeta) \otimes V_q(\Lambda_\xi)$ such that

$$\overline{P}_k(w_{k'}^{(\xi,\zeta)}) = \delta_{k,k'} w_k^{(\zeta,\xi)}. \tag{5.16}$$

For $\mathbf{a}, \mathbf{b} \in \mathbb{C}(q)^\times$, we define a $\mathbb{C}(q)$-linear map $\overline{R}_{\xi,\zeta}(\mathbf{a}, \mathbf{b}) : V_q(\Lambda_\xi)_\mathbf{a} \otimes V_q(\Lambda_\zeta)_\mathbf{b} \longrightarrow V_q(\Lambda_\zeta)_\mathbf{b} \otimes V_q(\Lambda_\xi)_\mathbf{a}$ whereby

$$\overline{R}_{\xi,\zeta}(\mathbf{a}, \mathbf{b}) := \sum_{k=\max(0,\xi+\zeta-n-1)}^{\min(\xi,\zeta)} \overline{\rho}_k \overline{P}_k, \tag{5.17}$$

where

$$\frac{\overline{\rho}_{k-1}}{\overline{\rho}_k} := \frac{\mathbf{b}-q^{\xi+\zeta-2k+2}\mathbf{a}}{\mathbf{a}-q^{\xi+\zeta-2k+2}\mathbf{b}}, \quad \overline{\rho}_{\min(\xi,\zeta)} := 1.$$

We call $\overline{R}_{\xi,\zeta}(\mathbf{a},\mathbf{b})$ a *R-matrix* of $V_q(\Lambda_\xi)_{\mathbf{a}} \otimes V_q(\Lambda_\zeta)_{\mathbf{b}}$. From [DO], §2.3 (and (4.17) in this paper), we obtain the following theorem.

Theorem 5.6 ([DO], §2.3)**.** *$\overline{R}_{\xi,\zeta}(\mathbf{a},\mathbf{b})$ is a $\widetilde{U}_q$-homomorphism.*

We have

$$\overline{\rho}_k = \prod_{p=k+1}^{\min(\xi,\zeta)} \frac{\mathbf{b}-q^{\xi+\zeta-2k+2}\mathbf{a}}{\mathbf{a}-q^{\xi+\zeta-2k+2}\mathbf{b}} = \prod_{p=1}^{\min(\xi,\zeta)-k} \frac{\mathbf{b}-q^{2p+|\xi-\zeta|}\mathbf{a}}{\mathbf{a}-q^{2p+|\xi-\zeta|}\mathbf{b}}, \quad (p \mapsto \min(\xi,\zeta)+1-p),$$

$$\overline{R}_{\xi,\zeta}(\mathbf{a},\mathbf{b}) = \sum_{k=0}^{\min(\xi,\zeta,n+1-\xi,n+1-\zeta)} \overline{\rho}_{\min(\xi,\zeta)-k}\overline{P}_{\min(\xi,\zeta)-k}, \quad (k \mapsto \min(\xi,\zeta)-k).$$

We set

$$R_{\xi,\zeta}(\mathbf{a},\mathbf{b}) := (\prod_{p=1}^{\min(\xi,\zeta)} (\mathbf{a}-q^{2p+|\xi-\zeta|}\mathbf{b}))\overline{R}_{\xi,\zeta}(\mathbf{a},\mathbf{b}), \tag{5.18}$$

$$\rho_k := (\prod_{p=1}^{\min(\xi,\zeta)} (\mathbf{a}-q^{2p+|\xi-\zeta|}\mathbf{b}))\overline{\rho}_{\min(\xi,\zeta)-k}, \quad P_k := \overline{P}_{\min(\xi,\zeta)-k}. \tag{5.19}$$

Then we have

$$R_{\xi,\zeta}(\mathbf{a},\mathbf{b}) = \sum_{k=0}^{\min(\xi,\zeta,n+1-\xi,n+1-\zeta)} \rho_k P_k, \qquad \rho_k = (\prod_{p=1}^{k}(\mathbf{b}-q^{2p+|\xi-\zeta|}\mathbf{a}))(\prod_{p=k+1}^{\min(\xi,\zeta)} (\mathbf{a}-q^{2p+|\xi-\zeta|}\mathbf{b})). \tag{5.20}$$

5.3 Reducibility: the $\widetilde{U}_q$ case

Proposition 5.7. *Let $\xi,\zeta \in I$ and $\mathbf{a},\mathbf{b} \in \mathbb{C}(q)^\times$. If there exists a $1 \le p_0 \le \min(\xi,\zeta,n+1-\xi,n+1-\zeta)$ such that $\mathbf{b} = q^{2p_0+|\xi-\zeta|}\mathbf{a}$ or $q^{-(2p_0+|\xi-\zeta|)}\mathbf{a}$, then $V_q(\Lambda_\xi)_{\mathbf{a}} \otimes V_q(\Lambda_\zeta)_{\mathbf{b}}$ is a reducible representation of $\widetilde{U}_q$.*

Proof. We assume $\mathbf{b} = q^{2p_0+|\xi-\zeta|}\mathbf{a}$. We can prove the case of $\mathbf{b} = q^{-(2p_0+|\xi-\zeta|)}\mathbf{a}$ similarly. It is enough to prove that $R_{\xi,\zeta}(\mathbf{a},\mathbf{b})$ is neither an isomorphism nor a zero map. Since $\mathbf{b} = q^{2p_0+|\xi-\zeta|}\mathbf{a}$, for $p_0 \le k \le \min(\xi,\zeta,n+1-\xi,n+1-\zeta)$, we have $\rho_k = 0$. Hence $R_{\xi,\zeta}(\mathbf{a},\mathbf{b})$ is not an isomorphism.

Now we assume $R_{\xi,\zeta}(\mathbf{a},\mathbf{b}) = 0$. Then $\rho_k = 0$ for any $0 \le k \le p_0-1$. Thus, for any $0 \le k \le p_0-1$, there exists a $0 \le p^{'} \le p_0 - 1$ such that $\mathbf{b} = q^{2p^{'}+|\xi-\zeta|}\mathbf{a}$ or a $p_0 \le p^{''} \le \min(\xi,\zeta)$ such that $\mathbf{a} = q^{2p^{''}+|\xi-\zeta|}\mathbf{b}$. Since $\mathbf{b} = q^{2p_0+|\xi-\zeta|}\mathbf{a}$, we obtain

$$p = p^{'}, \quad \text{or} \quad p_0 + p^{''} + |\xi-\zeta| = 0.$$

However this condition never occurs. Therefore $R_{\xi,\zeta}(\mathbf{a},\mathbf{b})$ is not a zero map. □

5.4 Main theorem: the $\widetilde{U}_q$ case

Theorem 5.8. *Let $m \in \mathbb{N}$, $\xi_1,\cdots,\xi_m \in I$, and $\mathbf{a}_1,\cdots,\mathbf{a}_m \in \mathbb{C}(q)^\times$. The following conditions (a) and (b) are equivalent.*

(a) $\widetilde{V}_q(\pi_{\xi_1}^{\mathbf{a}_1}) \otimes \cdots \otimes \widetilde{V}_q(\pi_{\xi_m}^{\mathbf{a}_m})$ is an irreducible representation of $\widetilde{U}_q$.

(b) For any $1 \le k \ne k^{'} \le m$ and $1 \le t \le \min(\xi_k,\xi_{k^{'}},n+1-\xi_k,n+1-\xi_{k^{'}})$,

$$\frac{\mathbf{a}_{k^{'}}}{\mathbf{a}_k} \ne q^{\pm(2t+|\xi_k-\xi_{k^{'}}|)}.$$

Proof. (b) is equivalent to the following condition $(b)^{'}$:
$(b)^{'}$ For any $1 \le k \ne k^{'} \le m$ and $\max(\xi_k, \xi_{k'}) \le t \le \min(\xi_k + \xi_{k'} - 1, n)$,

$$\frac{\mathbf{a}_k^{'}}{\mathbf{a}_k} \ne q^{\pm(2t-\xi_k-\xi_{k'}+2)}.$$

Therefore this theorem follows from Corollary 5.5 and Proposition 5.7. □

6 Tensor product of the fundamental representations for the restricted quantum loop algebras

In the rest of this paper, we fix the following notations. Let l be an odd integer greater than 2, let ε be a primitive l-th root of unity, and let $\mathcal{A} := \mathbb{C}[t, t^{-1}]$ be the Laurent polynomial ring. We regard $\mathbb{C}$ as an $\mathcal{A}$-algebra by the following formula:

$$g(q).c := g(\varepsilon)c \quad \text{for} \quad g(q) \in \mathcal{A}, c \in \mathbb{C},$$

and denote it by $\mathbb{C}_\varepsilon$.

6.1 Definition of the restricted quantum loop algebras

For $i \in \widetilde{I}$ and $m \in \mathbb{N}$, let $E_i^{(m)}$ and $F_i^{(m)}$ be as in (3.1). Let $\widetilde{U}_{\mathcal{A}}^{\mathrm{res}}$ (resp. $U_{\mathcal{A}}^{\mathrm{res}}$) be the $\mathcal{A}$-subalgebra of $\widetilde{U}_q$ (resp. U_q) generated by $\{E_i^{(m)}, F_i^{(m)}, K_i^{\pm 1} \,|\, i \in \widetilde{I}\,(\text{resp. } i \in I), m \in \mathbb{N}\}$. For $i \in I$, $r \in \mathbb{Z}$, and $m \in \mathbb{N}$, we define

$$\left[\begin{array}{c} K_i; r \\ m \end{array}\right] := \prod_{p=1}^{m} \frac{K_i q^{r-p+1} - K_i^{-1} q^{-r+p-1}}{q^p - q^{-p}}.$$

It is known that $\left[\begin{array}{c} K_i; r \\ m \end{array}\right] \in U_{\mathcal{A}}^{\mathrm{res}}$ (see [CP94b], §9.3A). Moreover, we have

$$(X_{i,r}^{\pm})^{(m)} := \frac{1}{[m]_q!}(X_{i,r}^{\pm})^m, \quad \frac{1}{[s]_q} H_{i,s}, \quad \mathcal{P}_{i,r} \in \widetilde{U}_{\mathcal{A}}^{\mathrm{res}},$$

for $i \in I$, $r \in \mathbb{Z}$, and $s \in \mathbb{Z}^{\times}$, where $\mathcal{P}_{i,r}$ be as in (3.6) (see [CP97], §3.1).

Definition 6.1. We define

$$\widetilde{U}_{\varepsilon}^{\mathrm{res}} := \widetilde{U}_{\mathcal{A}}^{\mathrm{res}} \otimes_{\mathcal{A}} \mathbb{C}_\varepsilon, \quad (\text{resp.} \quad U_{\varepsilon}^{\mathrm{res}} := U_{\mathcal{A}}^{\mathrm{res}} \otimes_{\mathcal{A}} \mathbb{C}_\varepsilon),$$

which is called a *restricted quantum loop algebra* (or *quantum loop algebra of Lusztig type*) (resp. *restricted quantum algebra* (or *quantum algebra of Lusztig type*)) (see [L89] and [CP97]).

For $i \in \widetilde{I}$, $j \in I$, $r \in \mathbb{Z}$, $s \in \mathbb{Z}^{\times}$, and $m \in \mathbb{N}$, we set

$$e_i^{(m)} := E_i^{(m)} \otimes 1, \quad f_i^{(m)} := F_i^{(m)} \otimes 1, \quad k_i := K_i \otimes 1, \quad \left[\begin{array}{c} k_j; r \\ m \end{array}\right] := \left[\begin{array}{c} K_j; r \\ m \end{array}\right],$$
$$(x_{i,r}^{\pm})^{(m)} := (X_{i,r}^{\pm})^{(m)} \otimes 1, \quad h_{i,s} := H_{i,s} \otimes 1, \quad \psi_{i,r}^{\pm} = \Psi_{i,r}^{\pm} \otimes 1, \quad \mathfrak{p}_{i,r} := \mathcal{P}_{i,r} \otimes 1 \quad \in \widetilde{U}_{\varepsilon}^{\mathrm{res}}.$$

6.2 The triangular decompositions of $U_{\varepsilon}^{\mathrm{res}}$ and $\widetilde{U}_{\varepsilon}^{\mathrm{res}}$

In a similar way to the proof of Lemma 3.4 in [AN], we can prove the following lemma.

Lemma 6.2. *Let V be a free $\mathcal{A}$-module and let $\{v_j\}_{j \in J}$ be a $\mathcal{A}$-basis of V, where J is an index set. Then $\{v_j \otimes 1\}_{j \in J}$ is a $\mathbb{C}$-basis of $V \otimes_{\mathcal{A}} \mathbb{C}_\varepsilon$.*

Proof. It is enough to prove that $\{v_j \otimes 1\}_{j\in J}$ is $\mathbb{C}$-linearly independent in $V \otimes_{\mathcal{A}} \mathbb{C}_\varepsilon$. For $c_j \in \mathbb{C}$ $(j \in J)$, we assume $\sum_{j\in J} c_j(v_j \otimes 1) = 0$. Then we have $\sum_{j\in J} c_j v_j \in (q-\varepsilon)V$. Since V is generated by $\{v_j\}_{j\in J}$ as a $\mathcal{A}$-module, there exist $d_j \in \mathcal{A}$ $(j \in J)$ such that

$$\sum_{j\in J} c_j v_j = (q-\varepsilon)\sum_{j\in J} d_j v_j.$$

Since $\{v_j\}_{j\in J}$ is $\mathcal{A}$-linearly independent in V, for any $j \in J$, we have $c_j = (q-\varepsilon)d_j$. Hence there exist $m \in \mathbb{N}$ and $c_{j,k} \in \mathbb{C}$ $(-m \le k \le m)$ such that

$$c_j = (q-\varepsilon)\sum_{k=-m}^{m} c_{j,k} q^k.$$

Therefore we obtain $c_j = 0$ for all $j \in J$. □

Let $(\widetilde{U}_\varepsilon^{\mathrm{res}})^+$ (resp. $(\widetilde{U}_\varepsilon^{\mathrm{res}})^-$, $(\widetilde{U}_\varepsilon^{\mathrm{res}})^0$) be the $\mathbb{C}$-subalgebra of $\widetilde{U}_\varepsilon^{\mathrm{res}}$ generated by $\{e_i^{(m)} \mid i \in \widetilde{I}, m \in \mathbb{N}\}$ (resp. $\{f_i^{(m)} \mid i \in \widetilde{I}, m \in \mathbb{N}\}$, $\{\mathfrak{p}_{i,r}, k_i, \begin{bmatrix} k_i;0 \\ l \end{bmatrix} \mid i \in I, r \in \mathbb{Z}, m \in \mathbb{N}\}$). Similarly, let $(U_\varepsilon^{\mathrm{res}})^+$ (resp. $(U_\varepsilon^{\mathrm{res}})^-$, $(U_\varepsilon^{\mathrm{res}})^0$) be the $\mathbb{C}$-subalgebra of $U_\varepsilon^{\mathrm{res}}$ generated by $\{e_i^{(m)} \mid i \in I, m \in \mathbb{N}\}$ (resp. $\{f_i^{(m)} \mid i \in I, m \in \mathbb{N}\}$, $\{k_i, \begin{bmatrix} k_i;0 \\ l \end{bmatrix} \mid i \in I\}$). From [L90a], we obtain the triangular decomposition of $U_\varepsilon^{\mathrm{res}}$, that is, the multiplication defines an isomorphism of $\mathbb{C}$-vector spaces:

$$(U_\varepsilon^{\mathrm{res}})^- \otimes (U_\varepsilon^{\mathrm{res}})^0 \otimes (U_\varepsilon^{\mathrm{res}})^+ \xrightarrow{\sim} U_\varepsilon^{\mathrm{res}}, \tag{6.1}$$

(see also [CP94b], Proposition 9.3.3 and [CP97], §1). From [CP97], §6, we obtain

$$\widetilde{U}_{\mathcal{A}}^{\mathrm{res}} = (\widetilde{U}_{\mathcal{A}}^{\mathrm{res}})^-(\widetilde{U}_{\mathcal{A}}^{\mathrm{res}})^0(\widetilde{U}_{\mathcal{A}}^{\mathrm{res}})^+.$$

Hence from (3.7), we obtain the triangular decomposition of $\widetilde{U}_{\mathcal{A}}^{\mathrm{res}}$:

$$(\widetilde{U}_{\mathcal{A}}^{\mathrm{res}})^- \otimes (\widetilde{U}_{\mathcal{A}}^{\mathrm{res}})^0 \otimes (\widetilde{U}_{\mathcal{A}}^{\mathrm{res}})^+ \xrightarrow{\sim} \widetilde{U}_{\mathcal{A}}^{\mathrm{res}}. \tag{6.2}$$

Therefore, from Lemma 6.2, we obtain the triangular decomposition of $\widetilde{U}_\varepsilon^{\mathrm{res}}$:

$$(\widetilde{U}_\varepsilon^{\mathrm{res}})^- \otimes (\widetilde{U}_\varepsilon^{\mathrm{res}})^0 \otimes (\widetilde{U}_\varepsilon^{\mathrm{res}})^+ \xrightarrow{\sim} \widetilde{U}_\varepsilon^{\mathrm{res}}. \tag{6.3}$$

6.3 The comultiplication of $U_\varepsilon^{\mathrm{res}}$ and $\widetilde{U}_\varepsilon^{\mathrm{res}}$

For $i \in I$ and $m \in \mathbb{N}$, we have

$$\Delta_H(E_i^{(m)}) = \sum_{p=0}^{m} q^{p(m-p)} E_i^{(m-p)} K_i^p \otimes E_i^{(p)}, \quad \Delta_H(F_i^{(m)}) = \sum_{p=0}^{m} q^{p(m-p)} F_i^{(p)} \otimes F_i^{(m-p)} K_i^{-p},$$
$$\epsilon_H(E_i^{(m)}) = \epsilon_H(F_i^{(m)}) = 0,$$
$$S_H(E_i^{(m)}) = (-1)^m q^{m(m-1)} K_i^{-m} E_i^{(m)}, \quad S_H(F_i^{(m)}) = (-1)^m q^{-m(m-1)} F_i^{(m)} K_i^m,$$

(see [Ja], §3–4). Hence $\widetilde{U}_{\mathcal{A}}^{\mathrm{res}}$ (resp. $U_{\mathcal{A}}^{\mathrm{res}}$) is a Hopf subalgebra of $\widetilde{U}_q$ (resp. U_q). In particular, we can define Hopf algebra structures on $\widetilde{U}_\varepsilon^{\mathrm{res}}$ and $U_\varepsilon^{\mathrm{res}}$. The comultiplication Δ_H^{res}, counit $\epsilon_H^{\mathrm{res}}$, and antipode S_H^{res} of $\widetilde{U}_\varepsilon^{\mathrm{res}}$ (resp. $U_\varepsilon^{\mathrm{res}}$) are given by

$$\Delta_H^{\mathrm{res}} = \Delta_H \otimes 1, \quad \epsilon_H^{\mathrm{res}} = \epsilon_H \otimes 1, \quad S_H^{\mathrm{res}} = S_H \otimes 1.$$

We define

$$X_{\pm,\mathcal{A}}^{\mathrm{res}} := \sum_{j\in I, r\in\mathbb{Z}, m\in\mathbb{N}} \mathcal{A}(X_{j,r}^{\pm})^{(m)}, \quad X_{\pm}^{\mathrm{res}} := X_{\pm,\mathcal{A}}^{\mathrm{res}} \otimes 1 = \sum_{j\in I, r\in\mathbb{Z}, m\in\mathbb{N}} \mathbb{C}(x_{j,r}^{\pm})^{(m)}.$$

Lemma 6.3. *Let $i \in I$ and $r \in \mathbb{N}$. Modulo $\widetilde{U}_\varepsilon^{\mathrm{res}} \otimes \widetilde{U}_\varepsilon^{\mathrm{res}} X_+^{\mathrm{res}}$,*

$$\Delta_H^{\mathrm{res}}(x_{i,r}^-) = x_{i,r}^- \otimes k_i + 1 \otimes x_{i,r}^- + \sum_{p=1}^{r-1} x_{i,r-p}^- \otimes \psi_{i,p}^+.$$

Proof. It is enough to prove that for $i \in I$ and $r \in \mathbb{Z}$, modulo $\widetilde{U}_{\mathcal{A}}^{\mathrm{res}} \otimes \widetilde{U}_{\mathcal{A}}^{\mathrm{res}} X_{+,\mathcal{A}}^{\mathrm{res}}$,

$$\Delta_H(X_{i,r}^-) = X_{i,r}^- \otimes K_i + 1 \otimes X_{i,r}^- + \sum_{p=1}^{r-1} X_{i,r-p}^- \otimes \Psi_{i,p}^+.$$

From Proposition 3.4 (b), modulo $\widetilde{U}_q \otimes \widetilde{U}_q X_+$,

$$\Delta_H(X_{i,r}^-) = X_{i,r}^- \otimes K_i + 1 \otimes X_{i,r}^- + \sum_{p=1}^{r-1} X_{i,r-p}^- \otimes \Psi_{i,p}^+.$$

On the other hand, since $\widetilde{U}_{\mathcal{A}}^{\mathrm{res}}$ is a Hopf subalgebra of $\widetilde{U}_\varepsilon^{\mathrm{res}}$, we have

$$\Delta_H(X_{i,r}^-) \in \widetilde{U}_{\mathcal{A}}^{\mathrm{res}} \otimes \widetilde{U}_{\mathcal{A}}^{\mathrm{res}}.$$

So it is enough to prove that

$$(\widetilde{U}_q \otimes \widetilde{U}_q X_+) \cap (\widetilde{U}_{\mathcal{A}}^{\mathrm{res}} \otimes \widetilde{U}_{\mathcal{A}}^{\mathrm{res}}) = \widetilde{U}_{\mathcal{A}}^{\mathrm{res}} \otimes \widetilde{U}_{\mathcal{A}}^{\mathrm{res}} X_{+,\mathcal{A}}^{\mathrm{res}}.$$

This follows from (3.7) and (6.2). □

6.4 Representation theory of $U_\varepsilon^{\mathrm{res}}$ and $\widetilde{U}_\varepsilon^{\mathrm{res}}$

We define

$$P_l := \{\lambda = \sum_{i \in I} \lambda_i \Lambda_i \in P \,|\, 0 \leq \lambda_i \leq l-1 \text{ for any } i \in I\}. \tag{6.4}$$

For $\mu = \sum_{i \in I} \mu_i \Lambda_i \in P$, there exist $\mu^{(0)} = \sum_{i \in I} \mu_i^{(0)} \Lambda_i \in P_l$ and $\mu^{(1)} = \sum_{i \in I} \mu_i^{(1)} \Lambda_i \in P$ such that $\mu = \mu^{(0)} + l\mu^{(1)}$.

Definition 6.4. Let V be a representation of $\widetilde{U}_\varepsilon^{\mathrm{res}}$ (resp. $U_\varepsilon^{\mathrm{res}}$).

(i) Let $v \in V$. If $(x_{i,r}^+)^{(m)} v = 0$ for all $i \in I$, $r \in \mathbb{Z}$, and $m \in \mathbb{N}$ (resp. $e_i^{(m)} v = 0$ for all $i \in I$ and $m \in \mathbb{N}$), we call v a *pseudo-primitive vector* (resp. *primitive vector*) in V.

(ii) For $\mu \in P$, we define

$$V_\mu := \{v \in V \,|\, k_i v = \varepsilon^{\mu_i^{(0)}} v, \quad \begin{bmatrix} k_i; 0 \\ l \end{bmatrix} v = \mu_i^{(1)} v \quad \text{for all } i \in I\}.$$

If $V_\mu \neq 0$, we call V_μ a *weight space* of V. For $v \in V_\mu$, we call v a weight vector with weight μ and define $\mathrm{wt}(v) := \mu$.

(iii) For any $\mathbb{C}$-algebra homomorphism $\Lambda : (\widetilde{U}_\varepsilon^{\mathrm{res}})^0 \longrightarrow \mathbb{C}$, we define

$$V_\Lambda := \{v \in V \,|\, uv = \Lambda(u)v \quad \text{for all } u \in (\widetilde{U}_\varepsilon^{\mathrm{res}})^0\}.$$

If $V_\Lambda \neq 0$, we call V_Λ a *pseudo-weight space* of V. For $v \in V_\Lambda$, we call v a pseudo-weight vector with pseudo-weight Λ and define $\mathrm{pwt}(v) := \Lambda$.

(iv) Let $\Lambda : (\widetilde{U}_\varepsilon^{\mathrm{res}})^0 \longrightarrow \mathbb{C}$ be a $\mathbb{C}$-algebra homomorphism and λ be an element in P_+. If there exists a nonzero pseudo-primitive vector $v_\Lambda \in V_\Lambda$ (resp. primitive vector $v_\lambda \in V_\lambda$) such that $V = \widetilde{U}_\varepsilon^{\mathrm{res}} v_\Lambda$ (resp. $V = U_\varepsilon^{\mathrm{res}} v_\lambda$), we call V a *pseudo-highest weight representation* of $\widetilde{U}_\varepsilon^{\mathrm{res}}$ (resp. *highest-weight representation* of $U_\varepsilon^{\mathrm{res}}$) with the *pseudo-highest weight* Λ (resp. *highest weight* λ) generated by a *pseudo-highest weight vector* v_Λ (resp. *highest-weight vector* v_λ).

Let V be a representation of $\widetilde{U}_\varepsilon^{\mathrm{res}}$ (resp. $U_\varepsilon^{\mathrm{res}}$). We call V of *type* $\mathbf{1}$ if $k_i^l = 1$ on V for any $i \in I$. In general, finite-dimensional irreducible representations of $\widetilde{U}_\varepsilon^{\mathrm{res}}$ (resp. $U_\varepsilon^{\mathrm{res}}$) are classified into 2^n types according to $\{\sigma : Q \longrightarrow \{\pm 1\};$ group homomorphism$\}$. It is known that for any $\sigma : Q \longrightarrow \{\pm 1\}$, the category of finite-dimensional representations of $\widetilde{U}_\varepsilon^{\mathrm{res}}$ (resp. $U_\varepsilon^{\mathrm{res}}$) of type σ is essentially equivalent to the category of the finite-dimensional representations of $\widetilde{U}_\varepsilon^{\mathrm{res}}$ (resp. $U_\varepsilon^{\mathrm{res}}$) of type $\mathbf{1}$.

We define a set of polynomials $\mathbb{C}_0[t]$ whereby

$$\mathbb{C}_0[t] := \{\pi(t) \in \mathbb{C}[t] \,|\, \pi(0) = 1\}.$$

For $\pi = (\pi_i(t))_{i \in I} \in (\mathbb{C}_0[t])^n$ (resp. $\lambda \in P_+$), let v_π (resp. v_λ) be a pseudo-highest weight vector in $\widetilde{V}_q(\pi)$ (resp. highest-weight vector in $V_q(\lambda)$) (see §3.2). We define

$$\begin{aligned} \widetilde{V}_{\mathcal{A}}^{\mathrm{res}}(\pi) &:= \widetilde{U}_{\mathcal{A}}^{\mathrm{res}} v_\pi, \quad \widetilde{W}_\varepsilon^{\mathrm{res}}(\pi) := \widetilde{V}_{\mathcal{A}}^{\mathrm{res}}(\pi) \otimes_{\mathcal{A}} \mathbb{C}_\varepsilon, \\ V_{\mathcal{A}}^{\mathrm{res}}(\lambda) &:= U_{\mathcal{A}}^{\mathrm{res}} v_\lambda, \quad W_\varepsilon^{\mathrm{res}}(\lambda) := V_{\mathcal{A}}^{\mathrm{res}}(\lambda) \otimes_{\mathcal{A}} \mathbb{C}_\varepsilon. \end{aligned}$$

We have

$$\dim_{\mathbb{C}}(\widetilde{W}_\varepsilon^{\mathrm{res}}(\pi)) = \dim_{\mathbb{C}(q)}(\widetilde{V}_q(\pi)), \quad \dim_{\mathbb{C}}(W_\varepsilon^{\mathrm{res}}(\lambda)) = \dim_{\mathbb{C}(q)}(V_q(\lambda)),$$

(see [CP94b], Proposition 11.2.5). For any $\widetilde{U}_\varepsilon^{\mathrm{res}}$-representation (resp. $U_\varepsilon^{\mathrm{res}}$-representation) V, we have

$$(x_{i,r}^{\pm})^{(m)} V_\mu \subset V_{\mu \pm m\alpha_i}, \quad (\text{resp.} \quad e_i^{(m)} V_\mu \subset V_{\mu + m\alpha_i}, \quad f_i^{(m)} V_\mu \subset V_{\mu - m\alpha_i}), \tag{6.5}$$

where $i \in I$, $r \in \mathbb{Z}$, $m \in \mathbb{N}$, and $\mu \in P$. Hence, by using (6.3), we have the following proposition.

Proposition 6.5. *For any $\pi \in (\mathbb{C}_0[t])^n$ (resp. $\lambda \in P_+$), $\widetilde{W}_\varepsilon^{\mathrm{res}}(\pi)$ (resp. $W_\varepsilon^{\mathrm{res}}(\lambda)$) has a unique irreducible quotient $\widetilde{V}_\varepsilon^{\mathrm{res}}(\pi)$ (resp. $V_\varepsilon^{\mathrm{res}}(\lambda)$).*

For any $\pi = (\pi_i(t))_{i \in I} \in (\mathbb{C}_0[t])^n$, there exists a unique $\mathbb{C}$-algebra homomorphism $\Lambda_\pi^{\mathrm{res}} : (\widetilde{U}_\varepsilon^{\mathrm{res}})^0 \longrightarrow \mathbb{C}$ such that

$$\Lambda_\pi^{\mathrm{res}}(k_i^{\pm 1}) = \varepsilon^{\pm \deg \pi_i(t)}, \quad \sum_{m=1}^{\infty} \Lambda_\pi^{\mathrm{res}}(\mathfrak{p}_{i,\pm m}) t^m = \pi_i^{\pm}(t),$$

where $\pi_i^{\pm}(t)$ be as in (3.9).

For any pseudo-highest weight representation of $\widetilde{U}_\varepsilon^{\mathrm{res}}$ with the pseudo-highest weight $\Lambda_\pi^{\mathrm{res}}$, we simply call it a pseudo-highest representation of $\widetilde{U}_\varepsilon^{\mathrm{res}}$ with the pseudo-highest weight π.

Theorem 6.6 ([L89])**.** *For $\lambda \in P_+$, $V_\varepsilon^{\mathrm{res}}(\lambda)$ is a finite-dimensional irreducible highest-weight representation of $U_\varepsilon^{\mathrm{res}}$ with the highest weight λ of type $\mathbf{1}$. Conversely, for any finite-dimensional irreducible $U_\varepsilon^{\mathrm{res}}$-representation V of type $\mathbf{1}$, there exists a unique $\lambda \in P_+$ such that V is isomorphic to $V_\varepsilon^{\mathrm{res}}(\lambda)$ as a representation of $U_\varepsilon^{\mathrm{res}}$.*

Theorem 6.7 ([CP97], §8)**.** *For $\pi \in (\mathbb{C}_0[t])^n$, $\widetilde{V}_\varepsilon^{\mathrm{res}}(\pi)$ is a finite-dimensional irreducible pseudo-highest weight representation of $\widetilde{U}_\varepsilon^{\mathrm{res}}$ with the pseudo-highest weight π of type $\mathbf{1}$. Conversely, for any finite-dimensional irreducible $\widetilde{U}_\varepsilon^{\mathrm{res}}$-representation V of type $\mathbf{1}$, there exists a unique $\pi \in (\mathbb{C}_0[t])^n$ such that V is isomorphic to $\widetilde{V}_\varepsilon^{\mathrm{res}}(\pi)$ as a representation of $\widetilde{U}_\varepsilon^{\mathrm{res}}$.*

Proposition 6.8 ([CP97], Proposition 7.4)**.** *Let V (resp. $V^{'}$) be a pseudo-highest weight representation of $\widetilde{U}_\varepsilon^{\mathrm{res}}$ with the pseudo-highest weight π (resp. $\pi^{'}$) generated by a pseudo-highest weight vector v_π (resp. $v^{'}_{\pi'}$). Then $v_\pi \otimes v^{'}_{\pi'}$ is a pseudo-primitive vector with* $\mathrm{pwt}(v_\pi \otimes v^{'}_{\pi'}) = \Lambda_{\pi\pi'}^{\mathrm{res}}$.

From Proposition 8.3 in [CP97] and Proposition 6.8, we obtain the following corollary.

Corollary 6.9. *Let $\pi, \pi^{'} \in (\mathbb{C}_0[t])^n$ and let v_π (resp. $v^{'}_{\pi'}$) be a pseudo-highest weight vector in $\widetilde{V}_\varepsilon^{\mathrm{res}}(\pi)$ (resp. $\widetilde{V}_\varepsilon^{\mathrm{res}}(\pi^{'})$). $\widetilde{V}_\varepsilon^{\mathrm{res}}(\pi\pi^{'})$ is isomorphic to a quotient of the $\widetilde{U}_\varepsilon^{\mathrm{res}}$-subrepresentation of $\widetilde{V}_\varepsilon^{\mathrm{res}}(\pi) \otimes \widetilde{V}_\varepsilon^{\mathrm{res}}(\pi^{'})$ generated by $v_\pi \otimes v^{'}_{\pi'}$. In particular, if $\widetilde{V}_\varepsilon^{\mathrm{res}}(\pi) \otimes \widetilde{V}_\varepsilon^{\mathrm{res}}(\pi^{'})$ is irreducible, $\widetilde{V}_\varepsilon^{\mathrm{res}}(\pi) \otimes \widetilde{V}_\varepsilon^{\mathrm{res}}(\pi^{'})$ is isomorphic to $\widetilde{V}_\varepsilon^{\mathrm{res}}(\pi\pi^{'})$.*

6.5 The fundamental representations of $U_\varepsilon^{\mathrm{res}}$ and $\widetilde{U}_\varepsilon^{\mathrm{res}}$

For $\xi \in I$ and $\mathbf{a} \in \mathbb{C}^\times$, let $\pi_\xi^{\mathbf{a}} = (\pi_{\xi,j}^{\mathbf{a}}(t))_{j\in I} \in (\mathbb{C}_0[t])^n$ be as in (4.13). We call $\widetilde{V}_\varepsilon^{\mathrm{res}}(\pi_\xi^{\mathbf{a}})$ (resp. $V_\varepsilon^{\mathrm{res}}(\Lambda_\xi)$) a *fundamental representation* of $\widetilde{U}_\varepsilon^{\mathrm{res}}$ (resp. $U_\varepsilon^{\mathrm{res}}$). We have

$$V_{\mathcal{A}}^{\mathrm{res}}(\Lambda_\xi) = \bigoplus_{\mu\in\mathcal{W}\Lambda_\xi} \mathcal{A}z_\mu, \tag{6.6}$$

and $W_\varepsilon^{\mathrm{res}}(\Lambda_\xi) = V_{\mathcal{A}}^{\mathrm{res}}(\Lambda_\xi) \otimes_{\mathcal{A}} \mathbb{C}_\varepsilon$ is irreducible as a representation of $U_\varepsilon^{\mathrm{res}}$. So we identify $W_\varepsilon^{\mathrm{res}}(\Lambda_\xi)$ with $V_\varepsilon^{\mathrm{res}}(\Lambda_\xi)$. For $z \in V_{\mathcal{A}}^{\mathrm{res}}(\Lambda_\xi)$, we set

$$\overline{z} := z \otimes 1 \in V_\varepsilon^{\mathrm{res}}(\Lambda_\xi).$$

From (6.6) and Lemma 6.2, we have

$$V_\varepsilon^{\mathrm{res}}(\Lambda_\xi) = \bigoplus_{\mu\in\mathcal{W}\Lambda_\xi} \mathbb{C}\overline{z}_\mu. \tag{6.7}$$

For $\xi \in I$ and $\mathbf{a} \in \mathbb{C}^\times$, let $V_{\mathcal{A}}^{\mathrm{res}}(\Lambda_\xi)_{\mathbf{a}}$ be the $\widetilde{U}_{\mathcal{A}}^{\mathrm{res}}$-subrepresentation of $V_q(\Lambda_\xi)_{\mathbf{a}}$ generated by z_{Λ_ξ}. We define $V_\varepsilon^{\mathrm{res}}(\Lambda_\xi)_{\mathbf{a}} := V_{\mathcal{A}}^{\mathrm{res}}(\Lambda_\xi)_{\mathbf{a}} \otimes_{\mathcal{A}} \mathbb{C}_\varepsilon$. Then, as representations of $\widetilde{U}_\varepsilon^{\mathrm{res}}$, we have

$$\widetilde{V}_\varepsilon^{\mathrm{res}}(\pi_\xi^{\mathbf{a}}) \cong V_\varepsilon^{\mathrm{res}}(\Lambda_\xi)_{\mathbf{a}}. \tag{6.8}$$

From (4.16), for $i \in I$, we have

$$h_{i,1}\overline{z}_{\Lambda_\xi} = \Lambda^{\mathrm{res}}_{\pi_\xi^{\mathbf{a}}}(h_{i,1})\overline{z}_{\Lambda_\xi} = \mathbf{a}\varepsilon^{-1}\delta_{i,\xi}\overline{z}_{\Lambda_\xi} \quad \text{in} \quad V_\varepsilon^{\mathrm{res}}(\Lambda_\xi)_{\mathbf{a}}. \tag{6.9}$$

Moreover, from Proposition 4.10, we obtain the following proposition.

Proposition 6.10. *Let $\xi \in I$ and $\mathbf{a} \in \mathbb{C}^\times$. For $i, j \in I$ such that $i \le j$, let $\omega_{i,j}$ be as in (3.18) and let $\overline{z_{\Lambda_\xi}(\omega_{i,j})}$ be the extremal vector in $V_\varepsilon^{\mathrm{res}}(\Lambda_\xi)_{\mathbf{a}}$.*

(a) We have $\overline{z_{\Lambda_\xi}(\omega_{i,j})} = \overline{z_{\omega_{i,j}\Lambda_\xi}}$.

(b) We have

$$h_{i-1,1}\overline{z_{\Lambda_\xi}(\omega_{i,j})} = \mathbf{a}\varepsilon^{2j-i-\xi}\delta(j-i+2 \le \xi \le j)\overline{z_{\Lambda_\xi}(\omega_{i,j})}, \quad \textit{if } i \ne 1,$$
$$h_{j+1,1}\overline{z_{\Lambda_\xi}(\omega_{1,j})} = \mathbf{a}\varepsilon^{j-\xi}\delta(1 \le \xi \le j+1)\overline{z_{\Lambda_\xi}(\omega_{1,j})}.$$

From (3.14), we have a $\mathbb{C}$-algebra involution $\Omega^{\mathrm{res}} : \widetilde{U}_\varepsilon^{\mathrm{res}} \longrightarrow \widetilde{U}_\varepsilon^{\mathrm{res}}$ such that

$$\Omega^{\mathrm{res}}((x_{i,r}^{\pm})^{(m)}) = -(x_{i,-r}^{\mp})^{(m)}, \quad \Omega^{\mathrm{res}}(h_{i,s}) = -h_{i,-s}, \quad \Omega^{\mathrm{res}}(\psi_{i,r}^{\pm}) = \psi_{i,-r}^{\mp}, \quad \Omega^{\mathrm{res}}(k_i^{\pm 1}) = k_i^{\mp 1},$$

for any $i \in I$, $r \in \mathbb{Z}$, $s \in \mathbb{Z}^\times$, and $m \in \mathbb{N}$. From Proposition 4.9, we have the following proposition.

Proposition 6.11. *Let $\xi \in I$ and $\mathbf{a} \in \mathbb{C}^\times$.*

(a) There exists an integer $c \in \mathbb{Z}$ depending only $\mathfrak{sl}_{n+1}$ such that, as a representation of $\widetilde{U}_\varepsilon^{\mathrm{res}}$, $V_\varepsilon^{\mathrm{res}}(\Lambda_\xi)_{\mathbf{a}}^*$ is isomorphic to $V_\varepsilon^{\mathrm{res}}(\Lambda_{n-\xi+1})_{\varepsilon^c\mathbf{a}}$.

(b) There exists a nonzero complex number $\kappa \in \mathbb{C}^\times$ depending only $\mathfrak{sl}_{n+1}$ such that, as a representation of $\widetilde{U}_\varepsilon^{\mathrm{res}}$, $V_\varepsilon^{\mathrm{res}}(\Lambda_\xi)_{\mathbf{a}}^{\Omega^{\mathrm{res}}}$ is isomorphic to $V_\varepsilon^{\mathrm{res}}(\Lambda_{n-\xi+1})_{\varepsilon^2\kappa\mathbf{a}^{-1}}$. In particular, $\overline{z_{\omega_{1,n}\Lambda_\xi}}$ is a pseudo-highest weight vector in $V_\varepsilon^{\mathrm{res}}(\Lambda_\xi)_{\mathbf{a}}^{\Omega^{\mathrm{res}}}$.

By using Proposition 6.8 and Proposition 6.11 (b), in a similar way to the proof of Lemma 4.12, we obtain the following lemma.

Lemma 6.12. *Let $\xi \in I$, $\mathbf{a} \in \mathbb{C}^\times$, and $\pi \in (\mathbb{C}_0[t])^n$. Let V be a pseudo-highest weight representation of $\widetilde{U}_q$ with the pseudo-highest weight π and let v_π be a pseudo-highest weight vector in V. We assume $\overline{z_{\omega_{1,n}\Lambda_\xi}} \otimes v \in \widetilde{U}_\varepsilon^{\mathrm{res}}(\overline{z_{\Lambda_\xi}} \otimes v)$. Then $V_\varepsilon^{\mathrm{res}}(\Lambda_\xi)_{\mathbf{a}} \otimes V$ is a pseudo-highest weight representation of $\widetilde{U}_\varepsilon^{\mathrm{res}}$ with the pseudo-highest weight $\pi_\xi^{\mathbf{a}}\pi$.*

6.6 Irreducibility: the $\widetilde{U}_\varepsilon^{\mathrm{res}}$ case

For $r \in \mathbb{Z}$, $s \in \mathbb{Z}^\times$, and $m \in \mathbb{N}$, we denote $(x_{1,r}^{\pm})^{(m)}$ (resp. $h_{1,s}$, $k_1^{\pm 1}$) in $U_\varepsilon^{\mathrm{res}}(\widetilde{\mathfrak{sl}}_2)$ by $(x_r^{\pm})^{(m)}$ (resp. h_s, $k^{\pm 1}$) and e_1 (resp. f_1, $k_1^{\pm 1}$) in $U_\varepsilon^{\mathrm{res}}(\mathfrak{sl}_2)$ by e (resp. f, $k^{\pm 1}$). For $i \in I$, let $(\widetilde{U}_\varepsilon^{\mathrm{res}})^{(i)}$ (resp. $(U_\varepsilon^{\mathrm{res}})^{(i)}$) be the $\mathbb{C}$-subalgebra of $\widetilde{U}_\varepsilon^{\mathrm{res}}$ (resp. $U_\varepsilon^{\mathrm{res}}$) generated by $\{(x_{i,r}^{\pm})^{(m)}, k_i^{\pm 1} \mid r \in \mathbb{Z}, m \in \mathbb{N}\}$ (resp. $\{e_i^{(m)}, f_i^{(m)}, k_i^{\pm 1} \mid m \in \mathbb{N}\}$). There exist $\mathbb{C}$-algebra homomorphisms $\widetilde{\iota}^{\mathrm{res}} : U_\varepsilon^{\mathrm{res}}(\widetilde{\mathfrak{sl}}_2) \longrightarrow (\widetilde{U}_\varepsilon^{\mathrm{res}})^{(i)}$ and $\iota^{\mathrm{res}} : U_\varepsilon^{\mathrm{res}}(\mathfrak{sl}_2) \longrightarrow (U_\varepsilon^{\mathrm{res}})^{(i)}$ such that

$$\widetilde{\iota}^{\mathrm{res}}((x_r^{\pm})^{(m)}) = (x_{i,r}^{\pm})^{(m)}, \quad \widetilde{\iota}^{\mathrm{res}}(h_s) = h_{i,s}, \quad \widetilde{\iota}^{\mathrm{res}}(k^{\pm 1}) = k_i^{\pm 1},$$
$$\iota^{\mathrm{res}}(e^{(m)}) = e_i^{(m)}, \quad \iota^{\mathrm{res}}(f^{(m)}) = f_i^{(m)}, \quad \iota^{\mathrm{res}}(k^{\pm 1}) = k_i^{\pm 1},$$

(see §5.1). Hence, for any $(\widetilde{U}_\varepsilon^{\mathrm{res}})^{(i)}$-representation (resp. $(U_\varepsilon^{\mathrm{res}})^{(i)}$-representation) V, we can regard V as a $U_\varepsilon^{\mathrm{res}}(\widetilde{\mathfrak{sl}}_2)$-representation (resp. $U_\varepsilon^{\mathrm{res}}(\mathfrak{sl}_2)$-representation).

In a similar way to the proof of Lemma 5.1, we can prove the following lemma.

Lemma 6.13. *Let $\xi \in I$ and $\mathbf{a} \in \mathbb{C}^\times$. For any $i, j \in I$ such that $i \leq j$, let $\overline{z_{\omega_{i,j}\Lambda_\xi}}$ be the extremal vector in $V_\varepsilon^{\mathrm{res}}(\Lambda_\xi)_{\mathbf{a}}$. As representations of $U_\varepsilon^{\mathrm{res}}(\widetilde{\mathfrak{sl}}_2)$,*

$$(\widetilde{U}_\varepsilon^{\mathrm{res}})^{(i-1)}\overline{z_{\omega_{i,j}\Lambda_\xi}} \cong \begin{cases} V_\varepsilon^{\mathrm{res}}(1)_{\mathbf{a}\varepsilon^{2j-\xi-i+1}}, & \text{if} \quad j-i+2 \leq \xi \leq j, \\ V_\varepsilon^{\mathrm{res}}(0)_{\mathbf{a}}, & \text{otherwise}, \end{cases} \qquad \text{if} \quad i \neq 1,$$

$$(\widetilde{U}_\varepsilon^{\mathrm{res}})^{(j+1)}\overline{z_{\omega_{1,j}\Lambda_\xi}} \cong \begin{cases} V_\varepsilon^{\mathrm{res}}(1)_{\mathbf{a}\varepsilon^{j-\xi+1}}, & \text{if} \quad 1 \leq \xi \leq j+1, \\ V_\varepsilon^{\mathrm{res}}(0)_{\mathbf{a}}, & \text{otherwise}. \end{cases}$$

Theorem 6.14. *Let $m \in \mathbb{N}$, $\xi_1, \cdots, \xi_m \in I$, and $\mathbf{a}_1, \cdots, \mathbf{a}_m \in \mathbb{C}^\times$. We assume that for any $1 \leq k < k' \leq m$ and $\max(\xi_k, \xi_{k'}) \leq t \leq \min(\xi_k + \xi_{k'} - 1, n)$,*

$$\frac{\mathbf{a}_{k'}}{\mathbf{a}_k} \neq \varepsilon^{2t-\xi_k-\xi_{k'}+2}.$$

Then $V_\varepsilon^{\mathrm{res}}(\Lambda_{\xi_1})_{\mathbf{a}_1} \otimes \cdots \otimes V_\varepsilon^{\mathrm{res}}(\Lambda_{\xi_m})_{\mathbf{a}_m}$ is a pseudo-highest weight representation of $\widetilde{U}_\varepsilon^{\mathrm{res}}$ with the pseudo-highest weight $\pi_{\xi_1}^{\mathbf{a}_1} \cdots \pi_{\xi_m}^{\mathbf{a}_m}$ generated by a pseudo-highest weight vector $\overline{z_{\Lambda_{\xi_1}}} \otimes \cdots \otimes \overline{z_{\Lambda_{\xi_m}}}$.

Proof. We can prove this theorem in a similar way to the proof of Theorem 5.3.

From Proposition 6.8, it is enough to prove

$$V_\varepsilon^{\mathrm{res}}(\Lambda_{\xi_1})_{\mathbf{a}_1} \otimes \cdots \otimes V_\varepsilon^{\mathrm{res}}(\Lambda_{\xi_m})_{\mathbf{a}_m} = \widetilde{U}_\varepsilon^{\mathrm{res}}(\overline{z_{\Lambda_{\xi_1}}} \otimes \cdots \otimes \overline{z_{\Lambda_{\xi_m}}}).$$

We shall prove this claim by the induction on m. If $m = 1$, we have nothing to prove. So we assume $m > 1$ and the case of $(m-1)$ holds. We set

$$V' := V_\varepsilon^{\mathrm{res}}(\Lambda_{\xi_2})_{\mathbf{a}_2} \otimes \cdots \otimes V_\varepsilon^{\mathrm{res}}(\Lambda_{\xi_m})_{\mathbf{a}_m}, \quad z' := \overline{z_{\Lambda_{\xi_2}}} \otimes \cdots \otimes \overline{z_{\Lambda_{\xi_m}}}.$$

From Proposition 6.8 and the assumption of the induction on m, V' is a pseudo-highest weight representation of $\widetilde{U}_\varepsilon^{\mathrm{res}}$ with the pseudo-highest weight $\pi_{\xi_2}^{\mathbf{a}_2} \cdots \pi_{\xi_m}^{\mathbf{a}_m}$ generated by a pseudo-highest weight vector z'. Hence, from Lemma 6.12, it is enough to prove that

$$\overline{z_{\omega_{1,n}\Lambda_{\xi_1}}} \otimes z' \in \widetilde{U}_\varepsilon^{\mathrm{res}}(\overline{z_{\Lambda_{\xi_1}}} \otimes z').$$

We shall prove that

$$\overline{z_{\omega_{i,j}\Lambda_{\xi_1}}} \otimes z' \in \widetilde{U}_\varepsilon^{\mathrm{res}}(\overline{z_{\Lambda_{\xi_1}}} \otimes z'), \tag{6.10}$$

for any $i, j \in I$ such that $i \leq j$. We define a total order in $I^{\leq}$ as (5.5). We shall prove (6.10) by the induction on (i, j). If $(i, j) = (1, 0)$, we have nothing to prove. So we assume that the case of (i, j) holds. We also assume $i \neq 1$. We can prove the case of $i = 1$ similarly. If $\xi_1 < j - i + 2$ or $\xi_1 > j$, we have

$$\overline{z_{\omega_{i-1,j}\Lambda_{\xi_1}}} \otimes z' = \overline{z_{\omega_{i,j}\Lambda_{\xi_1}}} \otimes z' \in \widetilde{U}_\varepsilon^{\mathrm{res}}(\overline{z_{\Lambda_{\xi_1}}} \otimes z').$$

So we assume $j-i+2 \le \xi_1 \le j$. From Lemma 6.3, for $r \in \mathbb{N}$, we have

$$\Delta_H^{\mathrm{res}}(x_{i-1,r}^-)(\overline{z_{\omega_{i,j}\Lambda_{\xi_1}}} \otimes z^{'}) - \overline{z_{\omega_{i,j}\Lambda_{\xi_1}}} \otimes (\Delta_H^{\mathrm{res}}(x_{i-1,r}^-)z^{'})$$
$$= (\Delta_H^{\mathrm{res}}(x_{i-1,r-k}^-)\overline{z_{\omega_{i,j}\Lambda_{\xi_1}}}) \otimes (\Delta_H^{\mathrm{res}}(k_{i-1})z^{'}) + \sum_{k=1}^{r-1}(x_{i-1,r-k}^-\overline{z_{\omega_{i,j}\Lambda_{\xi_1}}}) \otimes (\Delta_H^{\mathrm{res}}(\psi_{i-1,k}^+)z^{'}).$$

(see (5.6)). By using Lemma 6.13 and Lemma 5.2, we obtain

$$(\prod_{k\in M}(\mathbf{a}_k - \mathbf{a}_1\varepsilon^{2j-\xi_1-\xi_k+2}))(\prod_{k,k^{'}\in M, k<k^{'}}(\mathbf{a}_{k^{'}} - \mathbf{a}_k\varepsilon^2))(\overline{z_{\omega_{i-1,j}\Lambda_{\xi_1}}} \otimes z^{'}) \in \widetilde{U}_\varepsilon^{\mathrm{res}}(\overline{z_{\Lambda_{\xi_1}}} \otimes z^{'}),$$

where M be as in (5.13). Therefore, from the assumption of this theorem, we obtain

$$\overline{z_{\omega_{i-1,j}\Lambda_{\xi_1}}} \otimes z^{'} \in \widetilde{U}_\varepsilon^{\mathrm{res}}(\overline{z_{\Lambda_{\xi_1}}} \otimes z^{'}).$$

□

By using Theorem 6.14 and Proposition 6.11, we obtain the following corollary (see Corollary 5.5).

Corollary 6.15. *Let $m \in \mathbb{N}$, $\xi_1, \cdots, \xi_m \in I$, and $\mathbf{a}_1, \cdots, \mathbf{a}_m \in \mathbb{C}^\times$. We assume that for any $1 \le k \ne k^{'} \le m$ and $\max(\xi_k, \xi_{k^{'}}) \le t \le \min(\xi_k + \xi_{k^{'}} - 1, n)$,*

$$\frac{\mathbf{a}_{k^{'}}}{\mathbf{a}_k} \ne \varepsilon^{\pm(2t-\xi_k-\xi_{k^{'}}+2)}.$$

Then $V_\varepsilon^{\mathrm{res}}(\Lambda_{\xi_1})_{\mathbf{a}_1} \otimes \cdots \otimes V_\varepsilon^{\mathrm{res}}(\Lambda_{\xi_m})_{\mathbf{a}_m}$ is an irreducible representation of $\widetilde{U}_\varepsilon^{\mathrm{res}}$.

6.7 Reducibility: the $\widetilde{U}_\varepsilon^{\mathrm{res}}$ case

Proposition 6.16. *Let $\xi, \zeta \in I$ and $\mathbf{a}, \mathbf{b} \in \mathbb{C}^\times$. If there exists a $1 \le t \le \min(\xi, \zeta, n+1-\xi, n+1-\zeta)$ such that $\mathbf{b} = \varepsilon^{2t+|\xi-\zeta|}\mathbf{a}$ or $\varepsilon^{-(2t+|\xi-\zeta|)}\mathbf{a}$, then $V_\varepsilon^{\mathrm{res}}(\Lambda_\xi)_{\mathbf{a}} \otimes V_\varepsilon^{\mathrm{res}}(\Lambda_\zeta)_{\mathbf{b}}$ is reducible as a representation of $\widetilde{U}_\varepsilon^{\mathrm{res}}$.*

Proof. Let q be an indeterminate and let

$$(\mathbf{b}_q, \mathbf{b}_\varepsilon) = (q^{2t+|\xi-\zeta|}\mathbf{a}, \varepsilon^{2t+|\xi-\zeta|}\mathbf{a}) \quad \text{or} \quad (q^{-(2t+|\xi-\zeta|)}\mathbf{a}, \varepsilon^{-(2t+|\xi-\zeta|)}\mathbf{a}).$$

We assume that $V_\varepsilon^{\mathrm{res}}(\Lambda_\xi)_{\mathbf{a}} \otimes V_\varepsilon^{\mathrm{res}}(\Lambda_\zeta)_{\mathbf{b}_\varepsilon}$ is irreducible as a representation of $\widetilde{U}_\varepsilon^{\mathrm{res}}$. Since $V_\varepsilon^{\mathrm{res}}(\Lambda_\xi)_{\mathbf{a}} \cong \widetilde{V}_\varepsilon^{\mathrm{res}}(\pi_\xi^{\mathbf{a}})$ (resp. $V_\varepsilon^{\mathrm{res}}(\Lambda_\zeta)_{\mathbf{b}_\varepsilon} \cong \widetilde{V}_\varepsilon^{\mathrm{res}}(\pi_\zeta^{\mathbf{b}_\varepsilon})$), from Corollary 6.9, we obtain

$$V_\varepsilon^{\mathrm{res}}(\Lambda_\xi)_{\mathbf{a}} \otimes V_\varepsilon^{\mathrm{res}}(\Lambda_\zeta)_{\mathbf{b}_\varepsilon} \cong \widetilde{V}_\varepsilon^{\mathrm{res}}(\pi_\xi^{\mathbf{a}}\pi_\zeta^{\mathbf{b}_\varepsilon}).$$

Hence, from (6.7) and (6.8), we have

$$\dim_{\mathbb{C}}(\widetilde{V}_\varepsilon^{\mathrm{res}}(\pi_\xi^{\mathbf{a}}\pi_\zeta^{\mathbf{b}_\varepsilon})) = \dim_{\mathbb{C}}(V_\varepsilon^{\mathrm{res}}(\Lambda_\xi)_{\mathbf{a}}) \times \dim_{\mathbb{C}}(V_\varepsilon^{\mathrm{res}}(\Lambda_\zeta)_{\mathbf{b}_\varepsilon}) = \dim_{\mathbb{C}(q)}(V_q(\Lambda_\xi)_{\mathbf{a}}) \times \dim_{\mathbb{C}(q)}(V_q(\Lambda_\zeta)_{\mathbf{b}_q}).$$

On the other hand, by the definition of $\widetilde{V}_\varepsilon^{\mathrm{res}}(\pi_\xi^{\mathbf{a}}\pi_\zeta^{\mathbf{b}_\varepsilon})$, we have

$$\dim_{\mathbb{C}}(\widetilde{V}_\varepsilon^{\mathrm{res}}(\pi_\xi^{\mathbf{a}}\pi_\zeta^{\mathbf{b}_\varepsilon})) \le \dim_{\mathbb{C}(q)} \widetilde{V}_q(\pi_\xi^{\mathbf{a}}\pi_\zeta^{\mathbf{b}_q}),$$

(see §6.4). Thus, we have

$$\dim_{\mathbb{C}(q)}(V_q(\Lambda_\xi)_{\mathbf{a}}) \times \dim_{\mathbb{C}(q)}(V_q(\Lambda_\zeta)_{\mathbf{b}_q}) \le \dim_{\mathbb{C}(q)}(\widetilde{V}_q(\pi_\xi^{\mathbf{a}}\pi_\zeta^{\mathbf{b}_q})).$$

Hence, from Corollary 3.9,

$$V_q(\Lambda_\xi)_{\mathbf{a}} \otimes V_q(\Lambda_\zeta)_{\mathbf{b}_q} \cong \widetilde{V}_q(\pi_\xi^{\mathbf{a}}\pi_\zeta^{\mathbf{b}_q}).$$

In particular, $V_q(\Lambda_\xi)_{\mathbf{a}} \otimes V_q(\Lambda_\zeta)_{\mathbf{b}_q}$ is irreducible as a representation of $\widetilde{U}_q$. However, from Proposition 5.7, $V_q(\Lambda_\xi)_{\mathbf{a}} \otimes V_q(\Lambda_\zeta)_{\mathbf{b}_q}$ is reducible. This is absurd. Therefore $V_\varepsilon^{\mathrm{res}}(\Lambda_\xi)_{\mathbf{a}} \otimes V_\varepsilon^{\mathrm{res}}(\Lambda_\zeta)_{\mathbf{b}_\varepsilon}$ is reducible as a representation of $\widetilde{U}_\varepsilon^{\mathrm{res}}$. □

6.8 Main theorem: the $\widetilde{U}_{\varepsilon}^{\mathrm{res}}$ case

Theorem 6.17. *Let $m \in \mathbb{N}$, $\xi_1, \cdots, \xi_m \in I$, and $\mathbf{a}_1, \cdots, \mathbf{a}_m \in \mathbb{C}^{\times}$. The following conditions (a) and (b) are equivalent.*

(a) $\widetilde{V}_{\varepsilon}^{\mathrm{res}}(\pi_{\xi_1}^{\mathbf{a}_1}) \otimes \cdots \otimes \widetilde{V}_{\varepsilon}^{\mathrm{res}}(\pi_{\xi_m}^{\mathbf{a}_m})$ is an irreducible representation of $\widetilde{U}_{\varepsilon}^{\mathrm{res}}$.

(b) For any $1 \le k \neq k^{'} \le m$ and $1 \le t \le \min(\xi_k, \xi_{k'}, n+1-\xi_k, n+1-\xi_{k'})$,

$$\frac{\mathbf{a}_{k'}}{\mathbf{a}_k} \neq \varepsilon^{\pm(2t+|\xi_k-\xi_{k'}|)}.$$

Proof. This theorem follows from Corollary 6.15 and Proposition 6.16 (see Theorem 5.8). □

7 Tensor product of the fundamental representations for the small quantum loop algebras

7.1 The tensor product theorems

Let P_l be as in (6.4). For $\lambda \in P_+$, let $\lambda^{(0)} \in P_l$ and $\lambda^{(1)} \in P_+$ be as in §6.4.

Theorem 7.1 ([L89], Theorem 7.4)**.** *For $\lambda \in P_+$, $V_{\varepsilon}^{\mathrm{res}}(\lambda)$ is isomorphic to $V_{\varepsilon}^{\mathrm{res}}(\lambda^{(0)}) \otimes V_{\varepsilon}^{\mathrm{res}}(l\lambda^{(1)})$ as a representation of $U_{\varepsilon}^{\mathrm{res}}$.*

For $\pi(t) \in \mathbb{C}_0[t]$, we call $\pi(t)$ *l-acyclic* if it is not divisible by $(1-ct^l)$ for any $c \in \mathbb{C}^{\times}$ (see [FM], §2.6). We define

$$\mathbb{C}_l[t] := \{\pi(t) \in \mathbb{C}_0[t] \,|\, \pi(t) \text{ is } l\text{-acyclic}\}, \tag{7.1}$$

$$\mathbb{C}[t^l] := \{\pi(t) \in \mathbb{C}_0[t] \,|\, \text{there exists a polynomial } \pi^{'}(t) \in \mathbb{C}_0[t] \text{ such that } \pi(t) = \pi^{'}(t^l)\}. \tag{7.2}$$

For $\pi \in (\mathbb{C}_0[t])^n$, there exist unique $\pi^{(0)} = (\pi_i^{(0)}(t))_{i \in I} \in (\mathbb{C}_l[t])^n$ and $\pi^{(1)} = (\pi_i^{(1)}(t))_{i \in I} \in (\mathbb{C}_0[t^l])^n$ such that $\pi_i(t) = \pi_i^{(0)}(t)\pi_i^{(1)}(t)$ for any $i \in I$.

Theorem 7.2 ([CP97], Theorem 9.1)**.** *For $\pi \in (\mathbb{C}_0[t])^n$, $\widetilde{V}_{\varepsilon}^{\mathrm{res}}(\pi)$ is isomorphic to $\widetilde{V}_{\varepsilon}^{\mathrm{res}}(\pi^{(0)}) \otimes \widetilde{V}_{\varepsilon}^{\mathrm{res}}(\pi^{(1)})$ as a representation of $\widetilde{U}_{\varepsilon}^{\mathrm{res}}$.*

7.2 The Frobenius homomorphisms and the construction of $V_{\varepsilon}^{\mathrm{res}}(l\lambda)$ and $\widetilde{V}_{\varepsilon}^{\mathrm{res}}(\pi^{(1)})$

Let $\widetilde{U} := U(\widetilde{\mathfrak{sl}}_{n+1})$ (resp. $U := U(\mathfrak{sl}_{n+1})$) be the universal enveloping algebra of $\widetilde{\mathfrak{sl}}_{n+1}$ (resp. $\mathfrak{sl}_{n+1}$), that is, $\widetilde{U}$ (resp. U) is an associative algebra over $\mathbb{C}$ generated by $\{\bar{e}_i, \bar{f}_i, \bar{h}_i \,|\, i \in \widetilde{I} \text{ (resp. } i \in I)\}$ with the defining relations:

$$\bar{h}_i\bar{h}_j = \bar{h}_j\bar{h}_i, \quad \bar{h}_i\bar{e}_j - \bar{e}_j\bar{h}_i = \mathfrak{a}_{i,j}\bar{e}_j, \quad \bar{h}_i\bar{f}_j - \bar{f}_j\bar{h}_i = -\mathfrak{a}_{i,j}\bar{f}_j, \quad \bar{e}_i\bar{f}_j - \bar{f}_j\bar{e}_i = \delta_{i,j}\bar{h}_i,$$

$$\sum_{m=0}^{1-\mathfrak{a}_{i,j}} (-1)^m \frac{(1-\mathfrak{a}_{i,j})!}{m!(1-\mathfrak{a}_{i,j}-m)!}\bar{e}_i^m\bar{e}_j\bar{e}_i^{1-\mathfrak{a}_{i,j}-m} = \sum_{m=0}^{1-\mathfrak{a}_{i,j}} (-1)^m \frac{(1-\mathfrak{a}_{i,j})!}{m!(1-\mathfrak{a}_{i,j}-m)!}\bar{f}_i^m\bar{f}_j\bar{f}_i^{1-\mathfrak{a}_{i,j}-m} = 0 \quad i \neq j.$$

Then, from [CP97], §1, we have the following theorem (see also [CP94b], Theorem 9.3.12 and §11.2B).

Theorem 7.3 ([CP97], §1)**.** *There exist a $\mathbb{C}$-algebra homomorphism $\widetilde{\mathrm{Fr}}_{\varepsilon} : \widetilde{U}_{\varepsilon}^{\mathrm{res}} \longrightarrow \widetilde{U}$ such that*

$$\widetilde{\mathrm{Fr}}_{\varepsilon}(e_i^{(m)}) = \begin{cases} \frac{\bar{e}_i^{m/l}}{(m/l)!}, & \text{if } l \text{ divides } m, \\ 0, & \text{otherwise,} \end{cases} \quad \widetilde{\mathrm{Fr}}_{\varepsilon}(f_i^{(m)}) = \begin{cases} \frac{\bar{f}_i^{m/l}}{(m/l)!}, & \text{if } l \text{ divides } m, \\ 0, & \text{otherwise,} \end{cases}$$

$$\widetilde{\mathrm{Fr}}_{\varepsilon}(k_i) = 1, \quad \widetilde{\mathrm{Fr}}_{\varepsilon}([k_i; l]) = \bar{h}_i,$$

for any $i \in I$ and $m \in \mathbb{N}$.

For any $\widetilde{U}$-representation V, we can regard V as a $\widetilde{U}_\varepsilon^{\mathrm{res}}$-representation by using $\widetilde{\mathrm{Fr}}_\varepsilon$ and denote it by $\widetilde{\mathrm{Fr}}_\varepsilon^*(V)$. Similarly, for any U-representation V, we can regard V as a $U_\varepsilon^{\mathrm{res}}$-representation by using $\mathrm{Fr}_\varepsilon := \widetilde{\mathrm{Fr}}_\varepsilon|_{U_\varepsilon^{\mathrm{res}}} : U_\varepsilon^{\mathrm{res}} \longrightarrow U$ and denote it by $\mathrm{Fr}_\varepsilon^*(V)$. For $\pi \in (\mathbb{C}_0[t])^n$ (resp. $\lambda \in P_+$), let $\widetilde{V}(\pi)$ (resp. $V(\lambda)$) be the finite-dimensional irreducible representation of $\widetilde{U}$ (resp. U) with the pseudo-highest weight π (resp. highest-weight λ) (see [CP97], §2).

Theorem 7.4 ([L89], §7 and [CP94b], Proposition 11.2.11)**.** *For $\lambda \in P_+$, $V_\varepsilon^{\mathrm{res}}(l\lambda)$ is isomorphic to $\mathrm{Fr}_\varepsilon^*(V(\lambda))$ as a representation of $U_\varepsilon^{\mathrm{res}}$.*

Theorem 7.5 ([CP97], Theorem 9.3)**.** *For $\pi = (\pi_i(t))_{i\in I}, \pi' = (\pi_i'(t))_{i\in I} \in (\mathbb{C}_0[t])^n$ such that $\pi_i(t) = \pi_i'(t^l)$ for any $i \in I$, $\widetilde{V}_\varepsilon^{\mathrm{res}}(\pi)$ is isomorphic to $\widetilde{\mathrm{Fr}}_\varepsilon^*(V(\pi'))$ as a representation of $\widetilde{U}_\varepsilon^{\mathrm{res}}$.*

7.3 Definition and the representation theory of the small quantum algebras

Definition 7.6. Let $\widetilde{U}_\varepsilon^{\mathrm{fin}}$ (resp. $U_\varepsilon^{\mathrm{fin}}$) be the $\mathbb{C}$-subalgebra of $\widetilde{U}_\varepsilon^{\mathrm{res}}$ (resp. $U_\varepsilon^{\mathrm{res}}$) generated by $\{e_i, f_i, k_i \,|\, i \in \widetilde{I}\,(\text{resp.}\, i \in I)\}$. We call $\widetilde{U}_\varepsilon^{\mathrm{fin}}$ a *small quantum loop algebra* (resp. *small quantum algebra*).

For any $\widetilde{U}_\varepsilon^{\mathrm{fin}}$-representation (resp. $U_\varepsilon^{\mathrm{fin}}$-representation) V, we call V of *type* **1** if $k_i = 1$ on V for all $i \in I$. For $\pi \in (\mathbb{C}_0[t])^n$ (resp. $\lambda \in P_+$), we regard $\widetilde{V}_\varepsilon^{\mathrm{res}}(\pi)$ (resp. $V_\varepsilon^{\mathrm{res}}(\lambda)$) as a representation of $\widetilde{U}_\varepsilon^{\mathrm{fin}}$ (resp. $U_\varepsilon^{\mathrm{fin}}$) and denote it by $\widetilde{V}_\varepsilon^{\mathrm{fin}}(\pi)$ (resp. $V_\varepsilon^{\mathrm{fin}}(\lambda)$).

Theorem 7.7 ([L89], Proposition 7.1 and [CP94b], Proposition 11.2.10)**.** *For $\lambda \in P_l$, $V_\varepsilon^{\mathrm{fin}}(\lambda)$ is irreducible as a representation of $U_\varepsilon^{\mathrm{fin}}$. Moreover, for any finite-dimensional irreducible $U_\varepsilon^{\mathrm{fin}}$-representation V of type* **1***, there exists a unique $\lambda \in P_l$ such that V is isomorphic to $V_\varepsilon^{\mathrm{fin}}(\lambda)$ as a representation of $U_\varepsilon^{\mathrm{fin}}$.*

Theorem 7.8 ([CP97], Theorem 9.2 and [FM], Thorem 2.6)**.** *For $\pi \in (\mathbb{C}_l[t])^n$, $\widetilde{V}_\varepsilon^{\mathrm{fin}}(\pi)$ is irreducible as a representation of $\widetilde{U}_\varepsilon^{\mathrm{fin}}$. Moreover, for any finite-dimensional irreducible $\widetilde{U}_\varepsilon^{\mathrm{fin}}$-representation V of type* **1***, there exists a unique $\pi \in (\mathbb{C}_l[t])^n$ such that V is isomorphic to $\widetilde{V}_\varepsilon^{\mathrm{fin}}(\pi)$ as a representation of $\widetilde{U}_\varepsilon^{\mathrm{fin}}$.*

Remark 7.9. From Theorem 7.2 and Theorem 7.5 (resp. Theorem 7.1 and Theorem 7.4), in order to understand the finite-dimensional irreducible representations of $\widetilde{U}_\varepsilon^{\mathrm{res}}$ (resp. $U_\varepsilon^{\mathrm{res}}$), we may consider the one of $\widetilde{U}_\varepsilon^{\mathrm{fin}}$ (resp. $U_\varepsilon^{\mathrm{fin}}$).

7.4 Main theorem: the $\widetilde{U}_\varepsilon^{\mathrm{fin}}$ case

Theorem 7.10. *Let $m \in \mathbb{N}$, $\xi_1, \cdots, \xi_m \in I$, and $\mathbf{a}_1, \cdots, \mathbf{a}_m \in \mathbb{C}^\times$. The following conditions (a) and (b) are equivalent.*

(a) $\widetilde{V}_\varepsilon^{\mathrm{fin}}(\pi_{\xi_1}^{\mathbf{a}_1}) \otimes \cdots \otimes \widetilde{V}_\varepsilon^{\mathrm{fin}}(\pi_{\xi_m}^{\mathbf{a}_m})$ is an irreducible representation of $\widetilde{U}_\varepsilon^{\mathrm{fin}}$.

(b) For any $1 \le k \neq k' \le m$ and $1 \le t \le \min(\xi_k, \xi_{k'}, n+1-\xi_k, n+1-\xi_{k'})$,

$$\frac{\mathbf{a}_{k'}}{\mathbf{a}_k} \neq \varepsilon^{\pm(2t+|\xi_k-\xi_{k'}|)}.$$

Proof. If (b) does not hold, then (a) also does not hold from Theorem 6.17. So we assume that (b) holds. From Theorem 6.17 and Corollary 6.9, as representations of $\widetilde{U}_\varepsilon^{\mathrm{res}}$,

$$\widetilde{V}_\varepsilon^{\mathrm{fin}}(\pi_{\xi_1}^{\mathbf{a}_1}) \otimes \cdots \otimes \widetilde{V}_\varepsilon^{\mathrm{fin}}(\pi_{\xi_m}^{\mathbf{a}_m}) = \widetilde{V}_\varepsilon^{\mathrm{res}}(\pi_{\xi_1}^{\mathbf{a}_1}) \otimes \cdots \otimes \widetilde{V}_\varepsilon^{\mathrm{res}}(\pi_{\xi_m}^{\mathbf{a}_m}) \cong \widetilde{V}_\varepsilon^{\mathrm{res}}(\pi_{\xi_1}^{\mathbf{a}} \cdots \pi_{\xi_m}^{\mathbf{a}_m}).$$

Hence, from Theorem 7.8, it is enough to prove $\pi_{\xi_1}^{\mathbf{a}} \cdots \pi_{\xi_m}^{\mathbf{a}_m} \in (\mathbb{C}_l[t])^n$.

There exist $\pi_i(t) \in \mathbb{C}_0[t]$ $(i \in I)$ such that $\pi_{\xi_1}^{\mathbf{a}} \cdots \pi_{\xi_m}^{\mathbf{a}_m} = (\pi_i(t))_{i\in I}$. If there exists an index $i \in I$ such that $\pi_i(t) \notin \mathbb{C}_l[t]$, there exists a nonzero complex number c such that $(1-ct)(1-c\varepsilon t)\cdots(1-c\varepsilon^{l-1}t)$ divides $\pi_i(t)$. Then there exist $1 \le i_1, \cdots, i_t \le m$ such that

$$\xi_{i_1} = \cdots = \xi_{i_t}, \quad \mathbf{a}_{i_1} = c, \quad \mathbf{a}_{i_2} = c\varepsilon, \quad \cdots, \quad \mathbf{a}_{i_t} = c\varepsilon^{l-1}.$$

On the other hand, since (b) holds,

$$\frac{\mathbf{a}_{i_s}}{\mathbf{a}_{i_r}} \neq \varepsilon^{\pm 2},$$

for any $1 \leq r \neq s \leq t$. This is absurd. $\square$

Acknowledgements: I would like to thank Masaharu Kaneda, Hyohe Miyachi, Toshiki Nakashima, and Atsushi Nakayashiki for their helpful discussions.

References

[AK] T. Akasaka and M. Kashiwara, Finite-Dimensional Representations of Quantum affine algebras, Publ. RIMS, Kyoto Univ. **33** (1997), 839–867.

[AN] Y. Abe and T. Nakashima, Evaluation representations of quantum affine algebras at roots of unity, J. Math. Phys., **47** (2006), no. 8, 083514 1–28.

[B] J. Beck, Braid group action and quantum affine algebras, Comm. Math. Phys., **165** (1994), no. 3, 555–568.

[BCP] J. Beck, V. Chari, and A. Pressley, An algebraic characterization of the affine canonical basis, Duke Math. J., **99** (1999), no. 3, 455–487.

[BK] J. Beck and V. G. Kac, Finite-dimensional representations of quantum algebras, Amer. Math. Soc., **9** (1996), 391–423.

[C] V. Chari, Braid group actions and tensor product. Int. Math. Res. Not., **2002**, no. 7, 357–382.

[CP91] V. Chari and A. Pressley, Quantum Affine Algebras, Comm. Math. Phys., **142** (1991), 261–283.

[CP94a] V. Chari and A. Pressley, Small representations of quantum affine algebras, Lett. Math. Phys., **30** (1994), 131–145.

[CP94b] V. Chari and A. Pressley, A Guide to Quantum Groups, Cambridge University Press, Cambridge, 1994.

[CP95] V. Chari and A. Pressley, Quantum affine algebras and their representations, Amer. Math. Soc., **16** (1995), 59–78.

[CP96] V. Chari and A. Pressley, Minimal affinizations of representations of quantum groups: simply laced case, J. Algebra, **184** (1996), no. 1, 1–30.

[CP97] V. Chari and A. Pressley, Quantum algebras at roots of unity, Represent. Theory., **1** (1997), 280–328.

[CP01] V. Chari and A. Pressley, Weyl modules for classical and quantum affine algebras. Represent. Theory 5 (2001), 191–223.

[D] V. G. Drinfel'd, A new realization of Yangians and quantized affine algebras, Soviet Math. Dokl., **36** (1988), 212–216.

[DK] C. De Concini and V. G. Kac, Actes du Colloque en l'honneur de Jacques Diximier, edited by A. Connes, M. Duflo, A. Joseph and R. Rentschler (Prog. Math. Birkhauser), **92** (1990), 471–506.

[DO] E. Date and M. Okado, Calculation of excitation spectra of the spin model related with the vector representation of the quantized affine algebras of type $A_n^{(1)}$, Interna. J Modern Phys. A, **9** (1994), 399–417.

[FM] E. Frenkel and E. Mukhin, The q-characters at roots of unity, Adv. Math., **171** (2002), no. 1, 139–167.

[H] J. Humphreys, Introduction to Lie Algebras and Representation Theory, Graduate Texts in Mathematics, Vol. 9. Springer-Verlag, 1972.

[Ja] J. C. Jantzen, Lectures on Quantum Groups, Graduate Studies in Mathematics, 6, American Mathematical Society, 1996.

[Ji] M. Jimbo, A q-analogue of $U(\mathfrak{gl}(N+1))$, Heck algebra and Yang-Baxter equation, Lett. Math. Phys., **11** (1986), 247–252.

[Kac] V. G. Kac, Infinite dimensional Lie algebrs, 3rd edition, Cambridge Univ. Press, 1990.

[K] M. Kashiwara, On level-zero representations of quantized affine algebras, Duke Math. J., **112** (2002), no. 1, 117–175.

[L89] G. Lusztig, Modular representations and quantum groups, Contemp. Math., **82** (1989), 59–77.

[L90a] G. Lusztig, Finite dimensional Hopf algebras arising from quantized universal enveloping algebras, J. Amer. Math. Soc., **3** (1990), 257–296.

[L90b] G. Lusztig, Quantum groups at root of 1, Geom. Dedicata, **35** (1990), 89–113.

[L93] G. Lusztig, Introduction to Quantum Groups, Birkhäuser, Boston, 1993.

[VV] M. Varagnolo and E. Vasserot, Standard modules for quantum affine algebras, Duke Math. J., **111** (2002), no. 3, 509–533.